MATHEMATICAL CENTER IN AKADEMGORODOK

General constructions of biquandles and their symmetries

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Groups and quandles in low-dimensional topology
03.10.2020

Contents

V. Bardakov, T. Nasybullov, M. Singh, General constructions of biquandles and their symmetries, Arxiv:Math/1908.08301

- Definitions
- Motivation
- Fresh results
- Open problems

Quandles

Quandle Q is an algebraic system $(Q, *)$ such that

1. $x * x = x$ for all $x \in Q$
2. The map $S_x : y \mapsto y * x$ is a bijection of Q
3. $(x * y) * z = (x * z) * (y * z)$ for all $x, y, z \in Q$

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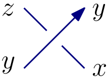
For a link L the quandle $Q(L)$ is the quandle with

Generators: labels on the arcs

Relations: $x * y = z$ near all crossings, where the labels are



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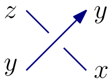

Theorem (Joyce 1982, Matveev 1982)

Two knot quandles $Q(K_1)$ and $Q(K_2)$ are isomorphic if and only if K_1 and K_2 are weakly equivalent

Quandle of a virtual link

For a virtual link L the quandle $Q(L)$ is the quandle with
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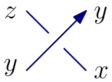

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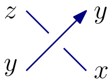

Theorem (Kauffman 1999)

The quandle $Q(L)$ is an invariant for virtual links

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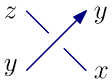

Theorem (Kauffman 1999)

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This invariant doesn't distinguish the virtual trefoil knot from the unknot

Biquandles

R. Fenn, M. Jordan-Santana, L. Kauffman, Biquandles and virtual links, Topology Appl., V. 145, N. 1-3, 2004, 157–175

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Biquandle B is an algebraic system $(B, \underline{*}, \overline{*})$ such that

1. $x \underline{*} x = x \overline{*} x$ for all $x \in B$,
2. the maps $\alpha_y, \beta_y : B \to B$ and $S : B \times B \to B \times B$ given by $\alpha_y(x) = x \underline{*} y$, $\beta_y(x) = x \overline{*} y$, $S(x, y) = (y \overline{*} x, x \underline{*} y)$ are bijections for all $y \in B$,
3. the equalities
    1. $(x \underline{*} y) \underline{*} (z \underline{*} y) = (x \underline{*} z) \underline{*} (y \overline{*} z)$,
    2. $(x \underline{*} y) \overline{*} (z \underline{*} y) = (x \overline{*} z) \underline{*} (y \overline{*} z)$,
    3. $(x \overline{*} y) \overline{*} (z \overline{*} y) = (x \overline{*} z) \overline{*} (y \underline{*} z)$

    hold for all $x, y, z \in B$

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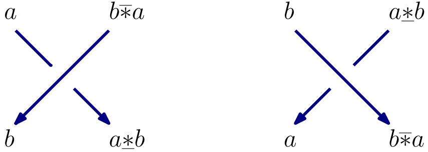
For a virtual link L the biquandle $B(L)$ is the biquandle with
Generators: labels on semiarcs
Relations:



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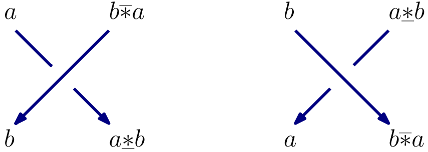

Theorem (Fenn-(Jordan-Santana)-Kauffman 2004)

The biquandle $B(L)$ is an invariant for virtual links

Biquandle of a virtual link

Conjecture (Fenn-(Jordan-Santana)-Kauffman 2004)

The biquandle $B(L)$ is an almost complete invariant for virtual links

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The number of homomorphisms from $B(L)$ to a given biquandle B is an invariant for virtual links

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The number of homomorphisms from $B(L)$ to a given biquandle B is an invariant for virtual links

Problem

Find canonical constructions of biquandles from quandles, groups, biquandles, etc

Biquandle on a union of quandles $Q_1 \sqcup Q_2$

Let $Q_1 = (X_1, *_1)$, $Q_2 = (X_2, *_2)$ be quandles

Biquandle on a union of quandles $Q_1 \sqcup Q_2$

Let $Q_1 = (X_1, *_1)$, $Q_2 = (X_2, *_2)$ be quandles, and let $\phi : Q_1 \to \mathrm{Conj}_{-1}(\mathrm{Aut}(Q_2))$ and $\psi : Q_2 \to \mathrm{Conj}_{-1}(\mathrm{Aut}(Q_1))$ be quandle homomorphisms such that

$$\phi_{x_1} = \phi_{\psi_{x_2}(x_1)}, \qquad \psi_{x_2} = \psi_{\phi_{x_1}(x_2)}$$

for all $x_1 \in Q_1$, $x_2 \in Q_2$.

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for all $x_1 \in Q_1$, $x_2 \in Q_2$.Then the set $X = X_1 \sqcup X_2$ with the operations $\overline{*}$, $\underline{*}$ given by

$$\begin{aligned}
a, b \in X_1 &\Rightarrow a \overline{*} b = a, a \underline{*} b = a *_1 b \\
a, b \in X_2 &\Rightarrow a \overline{*} b = a, a \underline{*} b = a *_2 b \\
a \in X_1, b \in X_2 &\Rightarrow a \overline{*} b = \psi_b(a), a \underline{*} b = \psi_b(a) \\
a \in X_2, b \in X_1 &\Rightarrow a \overline{*} b = \phi_b(a), a \underline{*} b = \phi_b(a)
\end{aligned}$$

is a biquandle

Biquandle on a product of quandles $Q_1 \times Q_2$

Let $Q_1 = (X_1, *_1)$, $Q_2 = (X_2, *_2)$ be quandles, and $\psi : Q_2 \to \mathrm{Conj}_{-1}(\mathrm{Aut}(Q_1))$ be a quandle homomorphism. Then the set $X_1 \times X_2$ with the operations

$$(x, a) \underline{*} (y, b) = \big(\psi_b(x *_1 y), a\big)$$
$$(x, a) \overline{*} (y, b) = \big(\psi_b(x), a *_2 b\big)$$

for $(x, a), (y, b) \in X_1 \times X_2$ is a biquandle.

Problems

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Specific problems:

- Find all biquandles on n elements such that $a \overline{*} b = a \underline{*} b$ for all a, b.
  W. Rump, A decomposition theorem for square-free unitary solutions of the quantum Yang-Baxter equation, Adv. Math., V. 193, N. 1, 2005, 40–55.
- Give a general definition of a semidirect product of (bi)quandles
  M. Castelli, F. Catino, P. Stefanelli, Left non-degenerate set-theoretic solutions of the Yang-Baxter equation and dynamical extensions of q-cycle sets, ArXiv:Math/2001.10774
- Does there exist an integer N such that every biquandle B of order at least N has a non-trivial automorphism?