Virtual twins and doodles

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Doodles on surfaces

- A *doodle* is represented by map $f : \sqcup_n S^1 \to \Sigma$ from n disjoint circles to a closed oriented surface such that $|f^{-1}f(x)| < 3$ for all $x \in \sqcup_n S^1$. That is, no triple or higher multiple points are created.
- Two such representatives are *equivalent* if they are equivalent under the equivalence generated by
  - (i) Homeomorphic equivalence
  - (ii) Homotopy through doodle representatives
  - (iii) Surface surgery disjoint from the diagram.

  This class is called a *doodle* or *doodle on a surface*.

Example

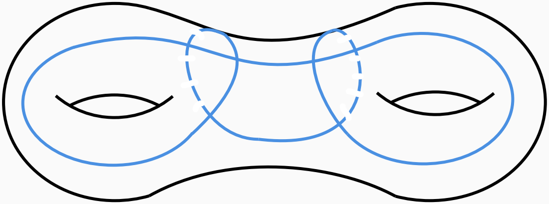


Figure 1: Kishino doodle on 2-torus

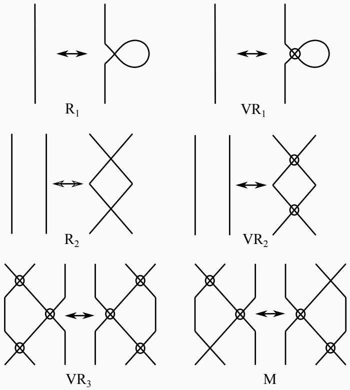
Virtual doodle diagram

- A *virtual doodle diagram* is a generic immersion of a closed one-dimensional manifold (disjoint union of circles) on the plane $\mathbb{R}^2$ with finitely many real or virtual crossings such that there are no triple or higher real intersection points.

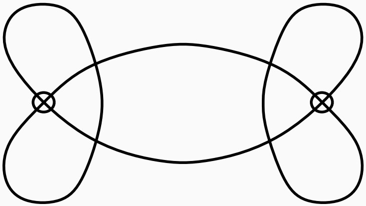
Figure 2: Real and virtual crossings

Equivalence of virtual doodle diagrams

Two virtual twin diagrams D_1 and D_2 on n strands are said to be *equivalent* if one can be obtained from the other by a finite sequence of moves (shown below) and isotopies of the plane.



Examples



There are so far

- 1 virtual doodle with 3 real crossings,
- 19 virtual doodle with 4 real crossings,
- 250 virtual doodle with 5 real crossings,
- 2477 virtual doodle with 6 real crossings,
- 2406 virtual doodle with 7 real crossings, and so on.

Relation between Doodles on surfaces and virtual doodles

Theorem (Bartholomew-Fenn-Kamada-Kamada)

There is a bijection between the family of oriented(unoriented) virtual doodles on plane and the family of oriented(unoriented) doodles on surfaces.

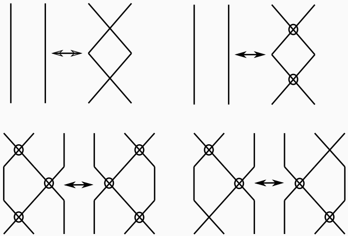
Virtual twins

A *virtual twin diagram* on n strands is a subset D of $\mathbb{R} \times [0,1]$ consists of n intervals with $\partial D = Q_n \times \{0,1\}$, Q_n is set of n points in $\mathbb{R}$ such that

1. the natural projection $\mathbb{R} \times [0,1] \to [0,1]$ maps each strand homeomorphically onto the unit interval $[0,1]$,
2. the set of all crossings of the diagram D consists of transverse double points of D where each crossing has the pre-assigned information of being a real or a virtual crossing.

Figure 3: Real and virtual crossings

Two virtual twin diagrams D_1 and D_2 on n strands are said to be *equivalent* if one can be obtained from the other by a finite sequence of moves and isotopies of the plane.



A *virtual twin* is an equivalence. The collection of virtual twins form a group with the operation of concatenation.

Abstract virtual twin group (Bardakov-Singh-Vesnin, 2019)

Consider group VT_n with generators $\{s_1, s_2, \ldots, s_{n-1}, \rho_1, \rho_2, \ldots, \rho_{n-1}\}$ and defining relations

$$\begin{aligned}
s_i^2 &= 1 \text{ for } i = 1, 2, \ldots, n-1, \\
s_i s_j &= s_j s_i \text{ for } |i-j| \geq 2, \\
\rho_i^2 &= 1 \text{ for } i = 1, 2, \ldots, n-1, \\
\rho_i \rho_j &= \rho_j \rho_i \text{ for } |i-j| \geq 2, \\
\rho_i \rho_{i+1} \rho_i &= \rho_{i+1} \rho_i \rho_{i+1} \text{ for } i = 1, 2, \ldots, n-2, \\
\rho_i s_j &= s_j \rho_i \text{ for } |i-j| \geq 2, \\
\rho_i \rho_{i+1} s_i &= s_{i+1} \rho_i \rho_{i+1} \text{ for } i = 1, 2, \ldots, n-2.
\end{aligned}$$

The kernel PVT_n of natural surjection of VT_n onto S_n is called *pure virtual twin group*.

Theorem (N., Singh, 2020)

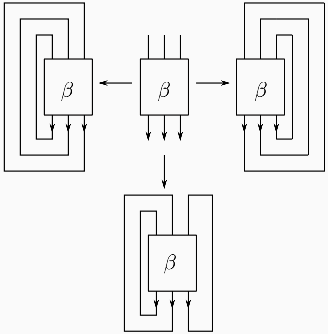
The group of virtual twins on n strands is isomorphic to VT_n.

Alexander theorem for virtual doodles

Theorem (N., Singh, 2020)

Every oriented virtual doodle on the plane is equivalent to closure of a virtual twin diagram.

By closure of a virtual twin diagram we mean a doodle obtained from the diagram by joining the end points with non-intersecting curves as shown below.



Markov theorem for virtual doodles

Theorem (N., Singh, 2020)

Two virtual twin diagrams on the plane (possibly on different number of strands) have equivalent closures if and only if they are related by a finite sequence of following moves.

($M0$) Defining relations 1 in VT_n,

($M1$) Conjugation: $\alpha^{-1}\beta\alpha \sim \beta$,

($M2$) Right stabilization of real or virtual type: $\beta s_n \sim \beta$ or $\beta\rho_n \sim \beta$,

($M3$) Left stabilization of real type: $(1 \otimes \beta)s_1 \sim \beta$,

($M4$) Right exchange: $\beta_1 s_n \beta_2 s_n \sim \beta_1 \rho_n \beta_2 \rho_n$,

($M5$) Left exchange: $s_1(1 \otimes \beta_1)s_1(1 \otimes \beta_2) \sim \rho_1(1 \otimes \beta_1)\rho_1(1 \otimes \beta_2)$.

Pictorial description of Markov moves

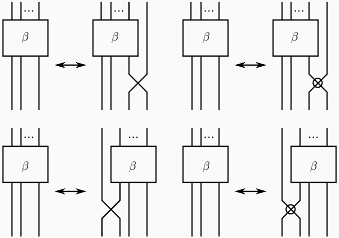


Figure 5: Left and right stabilisation of real and virtual type

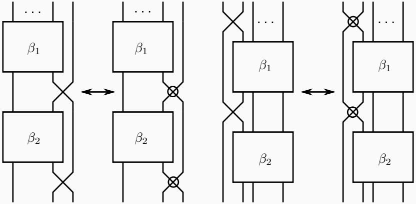


Figure 6: Left and right exchange

Structural properties of group VT_n and PVT_n (Naik, N., Singh, 2020)

Theorem

The pure virtual twin group PVT_n on $n \geq 2$ strands has the presentation

$$\langle \lambda_{i,j},\ 1 \leq i < j \leq n \mid \lambda_{i,j}\lambda_{k,l} = \lambda_{k,l}\lambda_{i,j} \text{ for distinct integers } i, j, k, l \rangle.$$

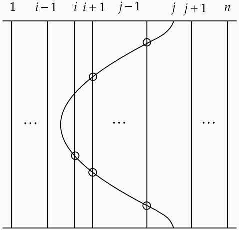


Figure 7: The generator $\lambda_{i,j}$

Structural properties of group VT_n and PVT_n (Naik, N., Singh, 2020)

Corollary

The pure virtual twin group PVT_n is an irreducible right-angled Artin group for each $n \geq 2$.

Corollary

The virtual twin group VT_n is residually finite and Hopfian for each $n \geq 2$.

Theorem

$\mathrm{Z}(PVT_n) = 1$ *for* $n \geq 3$ *and* $\mathrm{Z}(VT_n) = 1$ *for* $n \geq 2$.

Structural properties of group VT_n and PVT_n (Naik, N., Singh, 2020)

Theorem

For $n \geq 5$, $\mathrm{Aut}(PVT_n) \cong PVT_n \rtimes (\mathbb{Z}_2^{n(n-1)/2} \rtimes S_n)$.

Theorem

PVT_n has R_∞-property if and only if $n \geq 3$.

Theorem

$\gamma_2(VT_n) = \gamma_3(VT_n)$ for $n \geq 3$.

Corollary

VT_n is residually nilpotent if and only if $n = 2$.

Commutator subgroup of VT_n

Theorem

The commutator subgroup of the virtual twin group has the following presentation:

1. $\gamma_2(VT_2) \cong \mathbb{Z}$ *and is generated by* $(\rho_1 s_1)^2$.
2. $\gamma_2(VT_3) \cong \mathbb{Z}_3 * \mathbb{Z}_3 * \mathbb{Z}$ *and has a presentation*

$$\left\langle \rho_2\rho_1, s_1\rho_2\rho_1 s_1, (\rho_1 s_1)^2 \mid (\rho_2\rho_1)^3 = (s_1\rho_2\rho_1 s_1)^3 = 1 \right\rangle.$$

3. *For* $n \geq 4$, $\gamma_2(VT_n)$ *has a presentation with generators*

$$\left\{x_i, y_i, z \mid i = 2, 3, \ldots, n-1\right\}$$

and relations

$$
\begin{aligned}
x_2^3 &= 1, \\
x_j^2 &= 1 \quad \text{for } 3 \le j \le n-1, \\
y_2^3 &= 1, \\
(x_i x_{i+1}^{-1})^3 &= 1 \quad \text{for } 2 \le i \le n-2, \\
(x_i x_j^{-1})^2 &= 1 \quad \text{for } 2 \le i \le n-2 \text{ and } j \ge i+2, \\
(x_j z)^2 &= 1 \quad \text{for } 3 \le j \le n-1, \\
(y_2 z^{-1} x_3^{-1})^3 &= 1, \\
(y_2 z^{-1} x_j^{-1})^2 &= 1 \quad \text{for } 4 \le j \le n-1, \\
(y_2 z^{-1} x_3^{-1} y_2^{-1} x_2 x_3 x_2^{-1})^2 &= 1, \\
(z y_2^{-1} x_3 z y_2 z^{-1} x_2^{-1} x_3^{-1} x_2)^2 &= 1.
\end{aligned}
$$

References

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- Andrew Bartholomew, Roger Fenn, Naoko Kamada and Seiichi Kamada, *Doodles on surfaces*, J. Knot Theory Ramifications 27 (2018), no. 12, 1850071, 26 pp.
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- Tushar Kanta Naik, Neha Nanda and Mahender Singh, *Structural aspects of virtual twin groups*, (2020), arXiv:2008.10035.
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