Enumeration of coverings for closed orientable Euclidean manifolds

Alexander Mednykh
Sobolev Institute of Mathematics
Novosibirsk State University
Tomsk State University,

Tomsk, 03 October 2020

Definition. Consider a manifold $\mathcal{M}$. Two coverings $p_1 : \mathcal{M}_1 \to \mathcal{M}$ and $p_2 : \mathcal{M}_2 \to \mathcal{M}$ are said to be equivalent if there exists a homeomorphism $h : \mathcal{M}_1 \to \mathcal{M}_2$ such that $p_1 = p_2 \circ h$.

$$\begin{array}{ccc} \mathcal{M}_1 & \xrightarrow[homeo]{h} & \mathcal{M}_2 \\ p_1 \downarrow & & \downarrow p_2 \\ \mathcal{M} & \xrightarrow{id} & \mathcal{M} \end{array}$$

Coverings

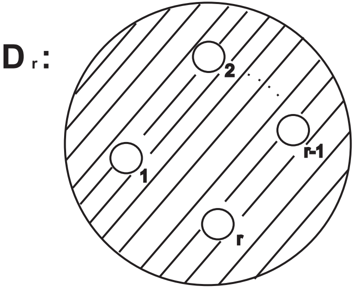
Let $p : \mathcal{M}_1 \to \mathcal{M}$ be a n-fold covering and $\Gamma = \pi_1(\mathcal{M})$ be the fundamental group of $\mathcal{M}$. Then there is an embedding

$$H_1 = \pi_1(\mathcal{M}_1) \underset{index\ n}{\subset} \Gamma = \pi_1(\mathcal{M}).$$

Two embeddings $H_1 = \pi_1(\mathcal{M}_1) \underset{n}{\subset} \Gamma$ and $H_2 = \pi_1(\mathcal{M}_2) \underset{n}{\subset} \Gamma$ produce *equivalent coverings* if and only if H_1 and H_2 *are conjugate* in Γ.

Consider the three classical cases.

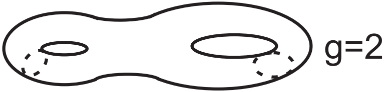
Case 1. Let S be a bordered surface of Euler characteristic $\chi = 1 - r,\ r \geq 0$. Than $\Gamma = \pi_1(S) \cong F_r$ is a free group of rank r. A typical example of S is the disc D_r with r holes removed:



Coverings

Case 2. Let S be a closed orientable surface of genus $g \geq 0$. Then

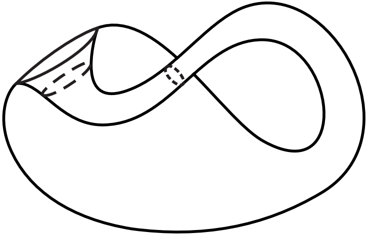
$$\pi_1(S) = \Phi_g = \langle a_1, b_1, \dots, a_g, b_g : \prod_{i=1}^{g} [a_i, b_i] = 1 \rangle$$



Coverings

Case 3. Let S be a closed non-orientable surface of genus $p \geq 1$.

$$\pi_1(S) = \Lambda_p = \langle a_1, a_2, \ldots, a_p : \prod_{i=1}^{p} a_i^2 = 1 \rangle$$



p = 2

Coverings

- Two main problems

From now on we deal with the following two problems.

Problem 1. Find the number $s_\Gamma(n)$ of subgroups of index n in the group Γ.

Problem 2. Find the number $c_\Gamma(n)$ of conjugacy classes of subgroups of index n in the group Γ.

Remark. In the latter case $c_\Gamma(n)$ coincides with the number of n-fold unbranched non-equivalent coverings of a manifold $\mathcal{M}$ with

$$\pi_1(\mathcal{M}) \equiv \Gamma.$$

Coverings

- Short history:

	Problem 1: $s_\Gamma(n)$	Problem 2: $c_\Gamma(n)$
1. $\Gamma = F_r$ $\Gamma = \pi_1(S),\ S = D_r$ bordered surface	M.Hall (1949)	V.Liskovets (1971) J.H.Kwak, J.Lee ($\geq$ 1971)
2. $\Gamma = \Phi_g$ $\Gamma = \pi_1(S),\ S = S_g$ orientable surface	A.Mednykh (1979)	A.Mednykh (1982)
3. $\Gamma = \Lambda_p$ $\Gamma = \pi_1(S),\ S = N_p$ non-orientable surface	G.Pozdnyakova, A.Mednykh (1986)	
4. $\Gamma = \pi_1(M)$, where M is a closed Seifert 3-manifold	V.Liskovets, M.(2000)	G.Chelnokov, M.Deryagina, M.(2016)

The present report is a part of the series of our papers devoted to enumeration of finite-sheeted coverings of coverings over closed Euclidean 3-manifolds. These manifolds are also known as flat 3-dimensional manifolds or just Euclidean 3-forms. The class of such manifolds is closely related to the notion of Bieberbach group. Recall that a subgroup of isometries of $\mathbb{R}^3$ is called Bieberbach group if it is discrete, cocompact and torsion free. Each 3-form can be represented as a quotient $\mathbb{R}^3/\Gamma$ where Γ is a Bieberbach group. In this case, Γ is isomorphic to the fundamental group of the manifold, that is $\Gamma = \pi_1(\mathbb{R}^3/\Gamma)$. Classification of three dimensional Euclidean forms up to homeomorphism is presented in the well-known monograph by J. Wolf. The class of such manifolds consists of six orientable $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4, \mathcal{G}_5, \mathcal{G}_6$, and four non-orientable ones $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4$.

In our previous paper [Deryagina-Chelnokov-M. 2017] we describe isomorphism types of finite index subgroups H in $\pi_1(\mathcal{B}_1)$ and $\pi_1(\mathcal{B}_2)$. Further, we calculate the respective numbers $s_{H,G}(n)$ and $c_{H,G}(n)$ for each isomorphism type H. The manifolds $\mathcal{B}_1$ and $\mathcal{B}_2$ are uniquely defined among the other non-orientable 3-forms by their homology groups $H1(\mathcal{B}_1) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $H_1(\mathcal{B}_2) = \mathbb{Z}_2$. In our paper [Chelnokov-M. 2020], similar questions were solved for manifolds $\mathcal{G}_2$ and $\mathcal{G}_4$ which are uniquely defined among flat compact 3-dimensional orientable manifolds without a boundary by their homology groups $H_1(\mathcal{G}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}$ and $H_1(\mathcal{G}_4) = \mathbb{Z}_2 \oplus \mathbb{Z}$. The aim of the present paper published in **Journal of Algebra 560 (2020) 48–66** is to investigate n-fold coverings over the orientable Euclidean manifolds $\mathcal{G}_3$ and $\mathcal{G}_5$, whose homology groups are $H_1(\mathcal{G}_3) = \mathbb{Z}_3 \oplus \mathbb{Z}$ and $H_1(\mathcal{G}_5) = \mathbb{Z}$.

To state the main result we need the following notations: $s_{H,G}(n)$ is the number of subgroups of index n in the group G, isomorphic to the group H; $c_{H,G}(n)$ is the number conjugacy classes of subgroups of index n in the group G, isomorphic to the group H. Also we will need the following combinatorial functions: In all cases we consider the function

$$\sigma_0(n) = \sum_{k|n} 1, \quad \sigma_1(n) = \sum_{k|n} k, \quad \sigma_2(n) = \sum_{k|n} \sigma_1(k), \quad \omega(n) = \sum_{k|n} k\sigma_1(k),$$

$$\theta(n) = |\{(p,q) \in \mathbb{Z}^2 | p > 0, q \geq 0, p^2 - pq + q^2 = n\}|.$$

Equivalently, $\theta(n) = \sum_{k|n} \frac{2}{\sqrt{3}} \sin \frac{2\pi k}{3}$. Also we suppose the above functions vanish if $n \notin \mathbb{N}$.

The first theorem provides the complete solution of the problem of enumeration of subgroups of a given finite index in $\pi_1(\mathcal{G}_3)$. For the sake of brevity, in case $H = \pi_1(\mathcal{G}_i)$ and $G = \pi_1(\mathcal{G}_j)$ we write $s_{i,j}$ and $c_{i,j}$ instead of $s_{H,G}(n)$ and $c_{H,G}(n)$ respectively.

Theorem

Every subgroup of finite index n in $\pi_1(\mathcal{G}_3)$ is isomorphic to either $\pi_1(\mathcal{G}_3)$ or $\pi_1(\mathcal{G}_1) = \mathbb{Z}^3$. The respective numbers of subgroups are

(i) $s_{3,3}(n) = \sum_{k|n} k\theta(k) - \sum_{k|\frac{n}{3}} k\theta(k)$,

(ii) $s_{1,3}(n) = \omega(\frac{n}{3})$.

The next theorem provides the number of conjugacy classes of subgroups of index n in $\pi_1(\mathcal{G}_3)$ for each isomorphism type. That is, the number of nonequivalent n-fold covering $\mathcal{G}_3$ which have a prescribe fundamental group.

Theorem

Let $\mathcal{N} \to \mathcal{G}_3$ be a n-fold covering over $\mathcal{G}_3$. If n is not divisible by 3 *then $\mathcal{N}$ is homeomorphic to $\mathcal{G}_3$. If n is divisible by* 3 *then $\mathcal{N}$ is homeomorphic to either $\mathcal{G}_3$ or $\mathcal{G}_1$. The corresponding numbers of nonequivalent coverings are given by the following formulas:*

(i) $c_{3,3}(n) = \sum_{k|n} \theta(k) + \sum_{k|\frac{n}{3}} \theta(k) - 2\sum_{k|\frac{n}{9}} \theta(k)$

(ii) $c_{1,3}(n) = \frac{1}{3}\left(\omega(\frac{n}{3}) + 2\sum_{k|\frac{n}{3}} \theta(k) + 4\sum_{k|\frac{n}{9}} \theta(k)\right)$

Appendix 2

Given a sequence $\{f(n)\}_{n=1}^{\infty}$, the formal power series

$$\widehat{f}(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

is called a Dirichlet generating function for $\{f(n)\}_{n=1}^{\infty}$. To reconstruct the sequence $f(n)$ from $\widehat{f}(s)$ one can use Perron's formula ([1], Th. 11.17). Given sequences $f(n)$ and $g(n)$ we call their *convolution* $(f * g)(n) = \sum_{k|n} f(k)g(\frac{n}{k})$. In terms of Dirichlet generating series the convolution of sequences corresponds to the multiplication of generating series $\widehat{f * g}(s) = \widehat{f}(s)\widehat{g}(s)$. For the above facts see, for example, ([1], Ch. 11–12).

Here we present the Dirichlet generating functions for the sequences $s_{H,G}(n)$ and $c_{H,G}(n)$. Since Theorems 1–4 provide the explicit formulas, the remainder is done by direct calculations.

By $\zeta(s)$ we denote the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$. Following [1] note that

$$\widehat{\sigma}_0(s) = \zeta^2(s), \quad \widehat{\sigma}_1(s) = \zeta(s)\zeta(s-1), \quad \widehat{\sigma}_2(s) = \zeta^2(s)\zeta(s-1),$$
$$\widehat{\omega}(s) = \zeta(s)\zeta(s-1)\zeta(s-2).$$

Define sequence $\{\chi(n)\}_{n=1}^{\infty}$ by $\chi(n) = \frac{2}{\sqrt{3}} \sin \frac{2\pi n}{3}$ or equivalently

$$\chi(n) = \begin{cases} 1 \text{ if } n \equiv 1 \mod 3 \\ -1 \text{ if } n \equiv 2 \mod 3 \\ 0 \text{ if } n \equiv 0 \mod 3. \end{cases}$$

For the sake of brevity denote $\vartheta(s) = \widehat{\chi}(s)$. Note that $\vartheta(s)$ is the Dirichlet L-series for the multiplicative character $\chi(n)$. Then $\widehat{\theta}(s) = \zeta(s)\vartheta(s)$. In more algebraic terms,

Table 2
Dirichlet generating functions for the sequences $s_{H,G}(n)$ and $c_{H,G}(n)$.

$H \backslash G$		$\pi_1(\mathcal{G}_3)$	$\pi_1(\mathcal{G}_5)$
$\pi_1(\mathcal{G}_1)$	$\widehat{s}_{H,G}$	$3^{-s}\zeta(s)\zeta(s-1)\zeta(s-2)$	$6^{-s}\zeta(s)\zeta(s-1)\zeta(s-2)$
	$\widehat{c}_{H,G}$	$3^{-s-1}\zeta(s)(\zeta(s-1)\zeta(s-2) + 2(1+2\cdot 3^{-s})\zeta(s)\vartheta(s))$	$6^{-s-1}\zeta(s)(\zeta(s-1)\zeta(s-2) + (1+3\cdot 2^{-s})\zeta(s)\zeta(s-1) + 4(1+3^{-s})\zeta(s)\vartheta(s))$
$\pi_1(\mathcal{G}_2)$	$\widehat{s}_{H,G}$	does not exist	$3^{-s}(1-2^{-s})\zeta(s)\zeta(s-1)\zeta(s-2)$
	$\widehat{c}_{H,G}$	does not exist	$3^{-s-1}(1-2^{-s})\zeta(s)^2\Big((1+3\cdot 2^{-s})\zeta(s-1) + 2\vartheta(s)\Big)$
$\pi_1(\mathcal{G}_3)$	$\widehat{s}_{H,G}$	$(1-3^{-s})\zeta(s)\zeta(s-1)\vartheta(s-1)$	$2^{-s}(1-3^{-s})\zeta(s)\zeta(s-1)\vartheta(s-1)$
	$\widehat{c}_{H,G}$	$(1-3^{-s})(1+2\cdot 3^{-s})\zeta(s)^2\vartheta(s)$	$2^{-s}(1-3^{-s})(1+3^{-s})\zeta(s)^2\vartheta(s)$
$\pi_1(\mathcal{G}_5)$	$\widehat{s}_{H,G}$	does not exist	$(1-2^{-s})(1-3^{-s})\zeta(s)\zeta(s-1)\vartheta(s-1)$
	$\widehat{c}_{H,G}$	does not exist	$(1-2^{-s})(1-3^{-s})\zeta(s)^2\vartheta(s)$