Discreteness Of Subgroups by Test Maps in Higher Dimension

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Groups and quandles in low-dimensional topology
3rd International Conference
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This talk is based on a work with Dr.Krishnendu Gongopadhyay, IISER(MOHALI) and Devendra Tiwary, Bhaskaracharya Pratisthana(Pune)

TITLE OF THE PRESENTATION

Discreteness Of Subgroups by Test Maps in Higher Dimension

Contents :

PLAN OF THE TALK

Definitions Needed..

Let..

$\mathbf{H}^n_{\mathbb{R}}$ denotes the n-dimensional (real) hyperbolic space and we denote $M(n)$ to be the (orientation-preserving) Möbius group that acts on $\mathbf{H}^{n+1}_{\mathbb{R}}$ by isometries. So in the previous case $M(2)$ can be identified with the linear group $\mathrm{SL}(2,\mathbb{C})$ that acts on the three dimensional hyperbolic space by the linear fractional transformations.

Generally speaking..

....$\mathbf{H}^n_{\mathbb{F}}$ be the n-dimensional hyperbolic space over $\mathbb{F}$, where $\mathbb{F}=\mathbb{R}$, $\mathbb{C}$ or the Hamilton's quaternions $\mathbb{H}$ Let $\mathrm{U}(n,1;\mathbb{F})$ the unitary group that acts on $\mathbf{H}^n_{\mathbb{F}}$ by isometries. For simplicity of notations, $\mathrm{U}(n,1;\mathbb{R})$ will be considered as the identity component of the full isometry group. Following standard notation, we denote $\mathrm{U}(n,1;\mathbb{R})=\mathrm{PO}(n,1)$.

Elementary Groups in $\mathrm{U}(n, 1; \mathbb{F})$..

Definition..

Let G be a subgroup of $\mathrm{U}(n, 1; \mathbb{F})$ and let $p \in \mathbf{H}^n_{\mathbb{R}} \cup \partial\mathbf{H}^n_{\mathbb{R}}$. Define $G_p = \{A \in G : A(p) = p\}$ to be the stabilizer of p in G. Also a $G-$orbit is a set defined by $G(p) = \{A(p) : A \in G\}$, for some $p \in \mathbf{H}^n_{\mathbb{R}} \cup \partial\mathbf{H}^n_{\mathbb{R}}$.
A subgroup G of $\mathrm{PSL}(2, \mathbb{C})$ is called **elementary** if $\mathbf{H}^n_{\mathbb{R}} \cup \partial\mathbf{H}^n_{\mathbb{R}}$ contains a finite $G-$orbit. Otherwise G is called **non-elementary**.

Basic Motivatory Results..

...

The familiar result that has been motivated us to study analogous aspects in our concerned group along this line of investigation is.. Let's have a look on this.

Jørgensen's Theorem on Discreteness in $\mathrm{SL}(2, \mathbb{C})$..

Jørgensen's Theorem on Discreteness in $\mathrm{SL}(2, \mathbb{C})$..

Jørgensen's Theorem on Discreteness in $\mathrm{SL}(2,\mathbb{C})$..

Significance of two-generator Subgroups: Jørgensen's Theorem..

A non-elementary group G of $\mathrm{PSL}(2,\mathbb{C})$ is discrete if and only if its two-generator subgroups are discrete .

Our Work...

...

Now we enter into the discussion on our results obtained so far. To begin with let us first depict some overviews and preliminaries to introduce the Hyperbolic $(n+2)-$space and its orientation-preserving isometry group $\mathrm{SL}(2, C_n)$.

MY PRIMARY AIM..

My primary aim in this talk is to focus on some interesting ideas to test discreteness of a Zariski dense subgroup G of $\mathrm{SL}(2, C_n)$, $n \geq 0$, which will be determined by two generator discrete subgroups in $\mathrm{SL}(2, C_n)$, having one isometry from the group and another suitable isometry from $\mathrm{SL}(2, C_n)$(need not be in the group) that influence the discreteness of the whole group in view of certain generalisations of Jørgensen inequality for two-generator subgroups in $\mathrm{SL}(2, C_n)$, that has been formulated and those results will be used in this work.

Preliminaries:- The Clifford Algebra

Definition

The *Clifford algebra* C_n, $n \geq 0$, **is the real associative algebra which has been generated by** n **symbols** $i_1, i_2, \cdots, i_n$ **subject to the following relations:**

$$i_t i_s = -i_s i_t \text{ , for } t \neq s \text{ and } i_t^2 = -1 \text{ .}$$

Let us define $i_0 = 1$ **and then every element of** C_n **can be expressed uniquely in the form** $a = \sum a_I I$ **, where the sum is over all products** $I = i_{v_1} i_{v_2} \cdots i_{v_k}$ **,with** $1 \leq v_1 < v_2 < \cdots < v_k \leq n$ **and** $a_I \in \mathbb{R}$**. Here the null product is permitted and identified with the real number** 1**. We equip** C_n **with the Euclidean norm. Thus** $C_0 = \mathbb{R}$, $C_1 = \mathbb{C}$, $C_2 = \mathbb{H}$ **etc.**

Important Involutions in C_n

Let us have a look..

The following are involutions in C_n:

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$*$: **In $a \in C_n$ as above, replace in each $I = i_{v_1} i_{v_2} \cdots i_{v_k}$ by $i_{v_k} \cdots i_{v_1}$. $a \mapsto a^*$ is an anti-automorphism.**

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$'$: Replace i_k by $-i_k$ in a to obtain a'.

The conjugate $\bar{a}$ of a is now defined as: $\bar{a} = (a^*)' = (a')^*$.

Some Definitions :

Clifford Vectors :

Let us identify $\mathbb{R}^{n+1}$ with the $(n+1)-$dimensional subspace of C_n formed by the Clifford numbers of the form $v = a_0 + a_1 i_1 + \ldots + a_n i_n$. These numbers are known as vectors. The products of non-zero vectors form a multiplicative group, denoted by Γ_n. For a vector v, $v^{-1} = \bar{v}/|v|^2$.

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Clifford Matrix..

A Clifford matrix of dimension n is a 2×2 matrix $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that
(i) a, b, c, $d \in \Gamma_n - \{0\}$;
(ii) the Clifford determinant $\Delta(T) = ad^* - bc^* = 1$, and,
(iii) ab^*, cd^*, c^*a, $d^*b \in \mathbb{R}^{n+1}$.

THE GROUP OF ISOMETRIES..

Definition

The group of all Clifford matrices is denoted by $\mathrm{SL}(2, C_n)$. In [Wat93], Waterman showed that $\mathrm{SL}(2, C_n)$ is same as the group of all invertible 2×2 matrices over C_n with Clifford determinant 1.

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Poincaré Extension...

The group $\mathrm{SL}(2, C_n)$ acts on $\mathbb{S}^{n+1} = \mathbb{R}^{n+1} \cup \{\infty\}$ by the action:

$$A : v \mapsto (av + b)(cv + d)^{-1}.$$

This action extends by Poincaré extension to $\mathbf{H}_{\mathbb{R}}^{n+2}$. The group $\mathrm{SL}(2, C_n)$ acts as the orientation-preserving isometry group of $\mathbf{H}_{\mathbb{R}}^{n+2}$.

The isometries of Hyperbolic $n-$Space....

...

One can classify the isometries of the space $\mathrm{H}_{\mathbb{R}}^{n+2}$ through the fixed points of the Möbius transformations in $\mathrm{SL}(2, C_n)$.

Characterization of isometries..

Let $A \in \mathrm{SL}(2, C_n)$ be a non-identity element. Then we define :

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(a) A is parabolic if A has exactly one fixed point on $\mathbb{R}^{\hat{n}+1}$.

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- **(a)** A is parabolic if A has exactly one fixed point on $\mathbb{R}^{\hat{n+1}}$.
- **(b)** A is elliptic if A has a fixed point in $\mathrm{H}_{\mathbb{R}}^{n+2}$.

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(b) A is elliptic if A has a fixed point in $\mathrm{H}_{\mathbb{R}}^{n+2}$.

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(c) A is loxodromic if A has two fixed points in $\mathbb{R}^{\hat{n}+1}$ and no fixed point in $\mathrm{H}_{\mathbb{R}}^{n+2}$.

(d) An elliptic element is called regular if it has a unique fixed point on $\mathrm{H}_{\mathbb{R}}^{n+2}$.

Classification of elements of $\mathrm{SL}(2, C_n)$:

Let us..

recall that a parabolic f element in $\mathrm{SL}(2, C_n)$ is conjugate to

$$\begin{pmatrix} \lambda & \mu \\ 0 & \lambda^{*-1} \end{pmatrix}, \ |\lambda| = 1, \ \mu \neq 0.$$

If $\lambda = 1$, then f is called a translation.

Up to conjugacy in $\mathrm{SL}(2, C_n)$, a loxodromic element f is given by

$$f = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{*-1} \end{pmatrix},$$

where $\lambda \in \Gamma_n$, $|\lambda| \neq 1$. If $|\lambda| = 1$, then it is a non-regular elliptic element.

Classification of elements of $\mathrm{SL}(2, C_n)$..

Regular Elliptic Isometry..

Suppose f is regular elliptic in $\mathrm{SL}(2, C_n)$, where n is even. Note that $\mathrm{SL}(2, C_n)$ has a natural inclusion in $\mathrm{SL}(2, C_{n+1})$ as a closed subgroup. We shall consider the inclusion of f in $\mathrm{SL}(2, C_{n+1})$, and assume that f fixes at least two points on the boundary $\partial \mathbf{H}_{\mathbb{R}}^{n+3}$. Otherwise, we can choose two fixed points of f on $\partial \mathbf{H}_{\mathbb{R}}^{n+2}$.

So, up to conjugacy in $\mathrm{SL}(2, C_{n+1})$, f is of the form

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{*-1} \end{pmatrix}, \ |\lambda| = 1.$$

The diagonal element λ depends on the rotation angles of f.

Important Definition..

Clifford Cross-Ratio..

Clifford cross ratios will be defined in an analogous way as like as in we do in Complex Analysis. Let $z_1, z_2, z_3, z_4 \in \partial \mathbf{H}_{\mathbb{R}}^{n+2}$ be any four distinct points. Let $z_1 \neq \infty$. The Clifford cross ratio of (z_1, z_2, z_3, z_4) is given by

$$\begin{aligned}
[z_1, z_2, z_3, z_4] &= (z_1 - z_3)(z_1 - z_2)^{-1}(z_2 - z_4)(z_3 - z_4)^{-1}, \\
& \quad \text{if } z_2, z_3, z_4 \neq \infty; \\
&= (z_1 - z_3)(z_3 - z_4)^{-1}, \text{ if } z_2 = \infty; \\
&= (z_1 - z_2)^{-1}(z_2 - z_4), \text{ if } z_3 = \infty; \\
&= (z_1 - z_3)(z - z_2)^{-1}, \text{ if } z_4 = \infty.
\end{aligned}$$

Properties of Cross-ratio..

..

One can easily prove that for any $f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, C_n)$, we have

$$[fz_1, fz_2, fz_3, fz_4] = (cz_3 + d)^{*-1}[z_1, z_2, z_3, z_4](cz_3 + d)^*.$$

Thus $|[z_1, z_2, z_3, z_4]|$ and $\mathrm{Re}[z_1, z_2, z_3, z_4]$ are invariants of Möbius maps in $\mathrm{SL}(2, C_n)$. We have the following basic properties of cross ratios.

1. $[z_1, z_2, z_3, z_4] + [z_2, z_1, z_3, z_4] = 1.$
2. $[z_1, z_2, z_3, z_4][z_4, z_2, z_3, z_1] = 1.$
3. $|[z_1, z_2, z_3, z_4]| = |[z_2, z_1, z_4, z_3]|.$
4. $|[z_1, z_2, z_3, z_4]| = |[z_3, z_4, z_1, z_2]|.$

Zariski-Dense Subgroups...

Definition: Zariski-Dense Subgroup..

Recall that a subgroup G of $\mathrm{SL}(2, C_n)$ is called *Zariski-dense* if it does not have a global fixed point and neither it preserves a proper totally geodesic subspace of $\mathbf{H}_{\mathbb{R}}^{(}n+2)$.

Our Goal : Problem

Problem :

Our goal is to show that a Zariski-dense subgroup G of $\mathrm{SL}(2, C_n)$ is discrete if for every loxodromic element $g \in G$, the two generator subgroup $\langle f, g\rangle$ is discrete, where $f \in \mathrm{SL}(2, C_n)$ is a certain element which is not necessarily from G.

Overviews

Motivation..

In this part of our discussion we try to test the discreteness of a zariski-dense subgroup of $\mathrm{SL}(2, C_n)$ using a test map not necessarily belongs to the given subgroup which is our prime interest. This line of investigation has been initially motivated by the remarkable result of Jørgensen and aftermath in higher dimensions Abikoff and Haas proved that a Zariski-dense subgroup G of $M(n)$ is discrete if and only if every two-generator subgroup $\langle f, g \rangle$ of G is discrete.

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Along this line Min Chen's Work..

..motivated us since he proved that a Zariski-dense subgroup G of $M(n)$ is discrete if the group $\langle f, g\rangle$ is discrete, where $g \in G$ and f is a fixed non-trivial element of $M(n)$(may be chosen outside G) which is not an irrational rotation i.e. of infinite order or if having finite order, does not pointwise fix the boundary.

Overviews

In Lower dimensions..

It is natural to ask whether the domain of the test map can be enlarged further. In low dimensions, the domain has been found to be much bigger. In [Yan09], [YZ14] and [Cao12], it is established that the discreteness of a non-elementary subgroup G of $\mathrm{SL}(2,\mathbb{C})$ is controlled by two-generator subgroups $\langle f, g\rangle$, where g is an element in G, and f is an element in $\mathrm{SL}(2,\mathbb{C})$. Thus any element from $\mathrm{SL}(2,\mathbb{C})$ may be chosen to be test map in this case, including irrational rotations.

In Higher dimensions..

..

One immediate generalization of the complex linear fractional transformations are the quaternionic linear fractional transformations that can be identified with the group $\mathrm{PSL}(2, \mathbb{H})$. Here $\mathrm{SL}(2, \mathbb{H})$, the 2×2 quaternionic matrices with Dieudonné determinant 1, acts by the linear fractional transformations on the boundary of the 5-dimensional hyperbolic space. In [GM17], also see [Kel03], some Jørgensen type inequalities for two generator subgroups of $\mathrm{SL}(2, \mathbb{H})$ were obtained. In [GMS18], these inequalities are used to prove that the discreteness of a Zariski-dense subgroup G of $\mathrm{SL}(2, \mathbb{H})$ is determined by two generator subgroups $\langle f, g \rangle$, where f is a certain test map from $\mathrm{SL}(2, \mathbb{H})$ and g is a loxodromic element of G. In particular it follows that certain irrational rotations may also be chosen as test maps.

Definition required..

The above results naturally raise the question whether the test maps may be chosen to be irrational rotations in higher dimensions where we shall adopt the Clifford algebraic formalism of $\mathrm{PO}(n,1)$ to answer this question by means Jørgensen type inequalities for two-generator subgroups of $\mathrm{SL}(2, C_n)$ in [Wat93]. Cao and Waterman extended Waterman's inequalities using conjugacy invariants in [CW98].

Given an isometry f of $\mathbf{H}_{\mathbb{R}}^{n+2}$, one can associate 'rotation angles' to it, and the rotation angles may be chosen to be elements of $(-\pi, \pi]$. The rotation angles are conjugacy invariants of an element, e.g. [Kul07]. One can further classify dynamical types of elements in $\mathrm{SL}(2, C_n)$ using the rotation angles and translation lengths, see [GK09]. For a non-elliptic isometry f, let τ_f denotes the translation length of f between the fixed points. $\tau_f = 0$ if and only if f is parabolic. The conjugacy invariant $\beta(f)$ used by Cao and Waterman can be defined in the next

Some Definitions..

Definition..

Let f be an element in $\mathrm{SL}(2, C_n)$. Let $\theta_1, \ldots, \theta_k \in (-\pi, \pi]$ be rotation angles of f (counted with multiplicities). Let $\Theta = \max_{1 \le i \le k} |\theta_i|$.
If f is elliptic or parabolic, then $\beta(f) = 4\sin^2(\Theta/2)$.
If f is loxodromic, then $\beta(f) = 4\sinh^2(\tau_f/2) + 4\sin^2(\Theta/2)$.

Cao-Waterman Inequalities..

We need call the following results which are important Jørgensen type inequalities for two-generator subgroups of $\mathrm{SL}(2, C_n)$ when one of the generators is either elliptic or hyperbolic.

....

Theorem

[CW98] Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, C_n)$ *be any element and* $f \in \mathrm{SL}(2, C_n)$ *be a loxodromic element having two fixed points* u, v *in* $\partial \mathbf{H}_{\mathbb{R}}^{n+2}$ *satisfying that* $\{gu, gv\}$ *is not equal to* $\{u, v\}$*. If* $\langle f, g \rangle$ *generate a discrete subgroup in* $\mathrm{SL}(2, C_n)$*, then*

$$\beta(f)\big(1 + |[u, v, gu, gv]|\big) \geq 1.$$

Cao-Waterman Inequalities..

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Theorem

[CW98] If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, C_n)$ *any element and* $f \in \mathrm{SL}(2, C_n)$ *be an elliptic element such that* $\langle f, g \rangle$ *forms a non-elementary discrete subgroup in* $\mathrm{SL}(2, C_n)$, *then we have*

$$\beta(f)\left(\frac{1}{4\sin^2(\pi/10)} + |[u, v, gu, gv]|\right) \geq 1,$$

where u, v *are any two boundary fixed points of* f.

Cao-Waterman Inequalities..

....

The Jørgensen type inequality for non-elliptic isometries fixing the boundary point ∞ is given by the following.

Theorem

[CW98] $f = \begin{pmatrix} \lambda & \mu \\ 0 & \lambda^{*-1} \end{pmatrix} \in \mathrm{SL}(2, C_n)$ *be a non-elliptic isometry that fixes the boundary point* ∞. *Let Let* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, C_n)$ *be any element in* $\mathrm{SL}(2, C_n)$ *such that* $0 < \rho = 2\cosh(\tau_f/2)\sqrt{\beta(f)} < 1$, *and* $fix(f) \cap fix(g) = \emptyset$. *If* $\langle f, g \rangle$ *generate a discrete subgroup in* $\mathrm{SL}(2, C_n)$, *then*

$$|tr^2(fgf^{-1})[fg(\infty), fg^{-1}(\infty), g(\infty), g^{-1}(\infty)]| \geq \frac{(1-\rho+\sqrt{(1-\rho)^2-4\beta(f)}}{2}.$$

Moreover, if f *is a translation, i.e.* $\lambda = 1$, *then we have*

$$|c|^2|\mu|^2 \geq \frac{(1-\rho+\sqrt{(1-\rho)^2-4\beta(f)}}{2}.$$

Some facts to be used

..

Let $\mathcal{L}$ be the set of loxodromic elements in $\mathrm{U}(n,1;\mathbb{F})$. It is well known that $\mathcal{L}$ is an open subset of $\mathrm{U}(n,1;\mathbb{F})$. This fact will be crucial for our proofs. Let $\mathcal{E}$ be the set of all regular elliptic elements in $\mathrm{U}(n,1;\mathbb{F})$. When $\mathbb{F}=\mathbb{C},\mathbb{H}$, $\mathcal{E}\neq\emptyset$. When $\mathbb{F}=\mathbb{R}$, note that $\mathcal{E}\neq\emptyset$ if and only if n is even. For n odd, an elliptic f in $\mathrm{U}(n,1;\mathbb{R})$ has at least two fixed points on $\partial\mathbf{H}^n_{\mathbb{R}}$. It is known that $\mathcal{E}$ is an open subset of $\mathrm{U}(n,1;\mathbb{F})$. The following theorem will also be useful for our purpose.

Theorem

Let G be a subgroup of $\mathrm{U}_0(n,1;\mathbb{F})$ such that there is no point in $\overline{\mathbf{H}^n_{\mathbb{F}}}$ or proper totally geodesic submanifold in $\mathbf{H}^n_{\mathbb{F}}$ which is invariant under G. Then G is either discrete or dense in $\mathrm{U}_0(n,1;\mathbb{F})$.

Some facts to be used

..

Let $L(G)$ be the limit set of a subgroup G of $\mathrm{U}(n,1;\mathbb{F})$. The limit set $L(G)$ is a closed G-invariant subset of $\partial\mathbf{H}^n_{\mathbb{F}}$. The group G is elementary if $L(G)$ is finite. If G is elementary, $L(G)$ consists of at most two points. If G is non-elementary, then $L(G)$ is an infinite set and every non-empty, closed G-invariant subset of $\partial\mathbf{H}^n_{\mathbb{F}}$ contains $L(G)$. We note the following lemma, for proof see [Rat06, Chapter 12].

Lemma

Let $a \in \partial\mathbf{H}^n_{\mathbb{F}}$ be fixed by a non-elliptic element of a subgroup G of $\mathrm{U}(n,1;\mathbb{F})$, then a is a limit point of G.

Main Theorems

We prove the following.

Main Theorem

[14] Let G be a Zariski-dense subgroup of $\mathrm{SL}(2, C_n)$.

1. Let f be a loxodromic element in $\mathrm{SL}(2, C_n)$, not necessarily in G, such that $0 < \beta(f) < 1$. If the two generator subgroup $\langle f, g\rangle$ is discrete for every loxodromic element g in G, then G is discrete.

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2. Let f be an elliptic element $\mathrm{SL}(2, C_n)$, not necessarily in G, such that $0 < \beta(f) < 4\sin^2(\frac{\pi}{10})$. If the two generator subgroup $\langle f, g \rangle$ is discrete for every loxodromic element g in G, then G is discrete.

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2. Let f be an elliptic element $\mathrm{SL}(2, C_n)$, not necessarily in G, such that $0 < \beta(f) < 4\sin^2(\frac{\pi}{10})$. If the two generator subgroup $\langle f, g \rangle$ is discrete for every loxodromic element g in G, then G is discrete.
3. Let f be a non-elliptic isometry in $\mathrm{SL}(2, C_n)$, not necessarily in G, such that
$$0 < \rho = 2\cosh(\tau_f/2)\sqrt{\beta(f)} < 1.$$
If the two generator subgroup $\langle f, g \rangle$ is discrete for every loxodromic element g in G, then G must be discrete.

PROOF

Sketch of the proof..

Let $Fix(f)$ be subset of $\overline{\mathbf{H}_{\mathbb{R}}}^{n+2}$ that is pointwise fixed by f. Let O_f be the stabilizer subgroup of $Fix(f)$ in $\mathrm{SL}(2, C_n)$. Clearly, O_f is a closed subgroup of $\mathrm{SL}(2, C_n)$. If possible suppose G is not discrete. Since G is Zariski-dense and assumed to be non-discrete, we must have that G is dense in $\mathrm{SL}(2, C_n)$. So $f \in \bar{G}$. Since $\mathcal{L} \cap (\mathrm{SL}(2, C_n) \setminus O_f)$ is an open subset of $\mathrm{SL}(2, C_n)$, there exists a sequence $\{g_n\}$ of loxodromic elements in $G \cap (\mathcal{L} \cap (\mathrm{SL}(2, C_n) \setminus O_f))$ such that $g_n \to f$.

(1) Without loss of generality, up to conjugacy, assume that $Fix(f) = \{0, \infty\}$, and

$$f = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{*-1} \end{pmatrix}, \ |\lambda| \neq 1.$$

PROOF

Sketch of the proof..

Note that the subgroups $\langle f, g_n \rangle$ are non-elementary, and they are discrete by hypothesis. Let $g_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$. It can be seen that $[0, \infty, g_n(0), g_n(\infty)] = -b_n c_n^*$. Thus using Cao-Waterman's inequality,

$$\beta(f)(1 + |b_n c_n|) \geq 1$$
$$\Rightarrow \quad |b_n c_n| \geq -1 + \frac{1}{\beta(f)} > 0.$$

But we have $b_n c_n \to 0$ as $n \to \infty$. This leads to a contradiction.

PROOF

Sketch of the proof..

(2) In this case, g_n and fg_nf^{-1} are elements in $\langle f, g_n\rangle$, so, by a known result, $\langle f, g_n\rangle$ is non-elementary. As in the above case, we have by Cao-Waterman's inequality, $|b_nc_n| \geq -\frac{1}{4\sin^2\frac{\pi}{10}} + \frac{1}{\beta(f)} > 0$, and we arrive at a contradiction.

(3) Applying suitable conjugation, without loss of generality we may assume that one of the fixed point of f be ∞ which leaves f in the form $f = \begin{pmatrix} \lambda & \mu \\ 0 & \lambda^{*-1} \end{pmatrix}$. Then using Cao-Waterman's inequality we must have $|tr^2(fg_nf^{-1})[fg_n(\infty), fg_n^{-1}(\infty), g_n(\infty), g_n^{-1}(\infty)]| \geq \frac{(1-\rho+\sqrt{(1-\rho)^2-4\beta(f)}}{2}$.

PROOF

Sketch of the proof..

By calculation, we see that the left hand side of the above inequality will be same as the left hand side of the following inequality:

$$|\lambda|^{-2}|c_n|^2|f(a_nc_n^{-1})-(a_nc_n^{-1})|.|f(-c_n^{-1}d_n)-(-c_n^{-1}d_n)| \geq \frac{(1-\rho+\sqrt{(1-\rho)^2-4\beta(f)}}{2},$$

i.e.

$$k_n = |c_n|^2|f(a_nc_n^{-1})-(a_nc_n^{-1})|.|f(-c_n^{-1}d_n)-(-c_n^{-1}d_n)| \geq \frac{|\lambda|^2(1-\rho+\sqrt{(1-\rho)^2-4\beta(f)}}{2}.$$

Since f and g_n does not have a common fixed point, we must have $c_n \neq 0$. Also since $0 < \rho < 1$, hence, $\frac{(1-\rho+\sqrt{(1-\rho)^2-4\beta(f)}}{2}$ is a positive real number. So, $|f(a_nc_n^{-1})-(a_nc_n^{-1})|$ and $|f(-c_n^{-1}d_n)-(-c_n^{-1}d_n)|$ are non-zero. Thus for all n, k_n is bounded above by a positive real number. But $k_n \to 0$ as $n \to \infty$. This is a contradiction. This proves the theorem.

IMPORTANT REMARK

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The main theorem says that one may choose certain irrational rotations as test maps which is a very interesting part of our investigation. If we go through very minutely then we may observe that our choice of test maps depends on the Cao-Waterman's inequalities to test our desired discreteness of G. After depicting the above theorem we have achieved another interesting result using similar way as in the main theorem .

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Another Discreteness Criterion

In this theorem we have achieved to show essentially that a subgroup G is discrete if every two generator subgroup of $\mathrm{SL}(2, C_n)$ is discrete where one generator is a test map f and the other is a conjugate of f by a loxodromic element of G and the complete statement is as follows :

DISCRETENESS CRITERION

Theorem

[15] Let G be a Zariski-dense subgroup of $\mathrm{SL}(2, C_n)$.

1. Let f be a loxodromic element in $\mathrm{SL}(2, C_n)$, not necessarily in G, such that $0 < \beta(f) < 1$. If the two generator subgroup $\langle f, gfg^{-1} \rangle$ is discrete for every loxodromic element g in G, then G is discrete.

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2. Let f be an elliptic element $\mathrm{SL}(2, C_n)$, not necessarily in G, such that $0 < \beta(f) < 4\sin^2(\frac{\pi}{10})$. If the two generator subgroup $\langle f, gfg^{-1} \rangle$ is non-elementary and discrete for every loxodromic element g in G, then G is discrete.

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2. Let f be an elliptic element $\mathrm{SL}(2, C_n)$, not necessarily in G, such that $0 < \beta(f) < 4\sin^2(\frac{\pi}{10})$. If the two generator subgroup $\langle f, gfg^{-1}\rangle$ is non-elementary and discrete for every loxodromic element g in G, then G is discrete.
3. Let f be a non-elliptic isometry in $\mathrm{SL}(2, C_n)$, not necessarily in G, such that
$$0 < \rho = 2\cosh(\tau_f/2)\sqrt{\beta(f)} < 1.$$
If the two generator subgroup $\langle f, gfg^{-1}\rangle$ is discrete for every loxodromic element g in G, then G must be discrete.

REMARK...

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The results in this paper show that in order to determine discreteness of a Zariski-dense subgroup G of $\mathrm{SL}(2, C_n)$, it is enough to check discreteness of the two generator subgroups of G obtained by adjoining the loxodromic elements of G to a 'test map' in $\mathrm{SL}(2, C_n)$, and further the test map may be chosen from an open subset of the isometry group. Let $\mathcal{E}$ denote the set of regular elliptic elements of $\mathrm{SL}(2, C_n)$. The set $\mathcal{E}$ is also an non-empty open subset of $\mathrm{SL}(2, C_n)$, provided n is even when $\mathbb{F} = \mathbb{R}$. Thus, if we replace the loxodromic elements g by regular elliptic elements, then our theorems go well for all even n.

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THANK YOU