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Cyclically presented groups and 3-manifolds

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Cyclically presented groups

Fibonacci groups

$$F(2, n) = \langle x_1, \ldots, x_n \quad | \quad x_i x_{i+1} = x_{i+2}, \quad i = 1, \ldots, n \rangle$$

were introduced by Conway.

He asked: whether or not $F(2,5)$ is a cyclic group of order 11?

More general question: whether or not groups $F(2,n)$ are infinite?

Conway [1967], Brunner [1974], Havas [1976], Chalk–Johnson [1976], Newman [1988, Thomas [1989]:

$F(2,n)$ is finite if and only if $n = 1, 2, 3, 4, 5, 7$.

More precisely, $F(2,3) = Q_8$, $F(2,4) = \mathbb{Z}_5$, $F(2,5) = \mathbb{Z}_{11}$, $F(2,7) = \mathbb{Z}_{29}$.

[1] J. Conway, Advanced problem 5327. Amer. Math. Monthly 72 (1965), 915.

Balanced presentations

A finite presentation $\langle X|R\rangle$ of a group G is called balanced if number of defining relations $|R|$ is equal to number of generators $|X|$.

Balanced presentations of groups appears very naturally in topology of 3-manifolds. In particular, Heegaard splitting of 3-manifolds induces balanced presentations of their fundamental groups.

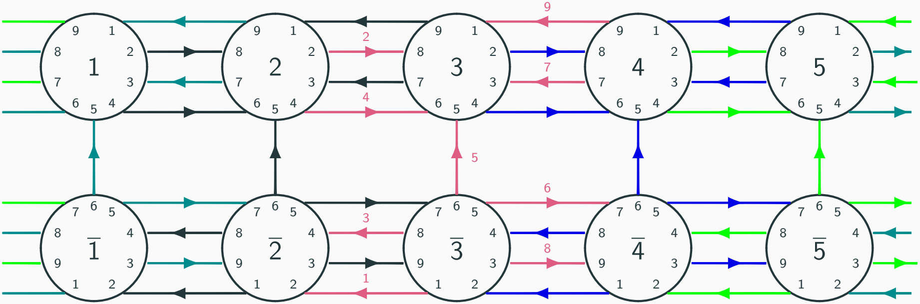
A group presentation called geometric if it corresponds a Heegaard splitting of a manifold.

Example. The balanced presentation

$$\langle a_1, a_2, a_3, a_4, a_5 \mid (a_{i-1}^{-1}a_i)^2 a_i (a_i^{-1}a_{i+1})^{-2} = 1, \quad i = 1,2,3,4,5\rangle$$

corresponds to the Heegaard splitting of a 3-manifold.

Example of balanced presentation



Heegaard diagram for related to the above balanched presentation.

The red curve is given by gluing:

$$\overline{3}_1 \longrightarrow \overline{2}_2 \xrightarrow{a_2^{-1}} 2_2 \longrightarrow 3_8 \xrightarrow{a_3} \overline{3}_8 \longrightarrow \overline{2}_4 \xrightarrow{a_2^{-1}} 2_4 \longrightarrow 3_6 \xrightarrow{a_3} \overline{3}_6 \longrightarrow 3_5$$

$$\xrightarrow{a_3} \overline{3}_5 \longrightarrow \overline{4}_7 \xrightarrow{a_4^{-1}} 4_7 \longrightarrow 3_3 \xrightarrow{a_3} \overline{3}_3 \longrightarrow \overline{4}_9 \xrightarrow{a_4^{-1}} 4_9 \longrightarrow 3_1 \xrightarrow{a_3} \overline{3}_1.$$

Cyclically presented groups

Let $\mathbb{F}_n$ be a free group of rank $n \geq 1$ with generators $x_1, x_2, \ldots, x_n$ and let $w = w(x_1, x_2, \ldots, x_n)$ be a cyclically reduced word in $\mathbb{F}_n$.

Let $\theta : \mathbb{F}_n \to \mathbb{F}_n$ be an automorphism given by $\theta(x_i) = x_{i+1}$, where $i = 1, \ldots, n-1$, and $\theta(x_n) = x_1$. The presentation

$$\mathcal{G}_n(w) = \langle x_1, \ldots, x_n \quad | \quad w = 1, \, \theta(w) = 1, \, \ldots, \, \theta^{n-1}(w) = 1 \rangle$$

is called an n-cyclic presentation with defining word w.

A group $\mathcal{G}$ is said to be cyclically presented if $\mathcal{G}$ is isomorphic to $\mathcal{G}_n(w)$ for some n and w.

Central problems

A group theory point of views:

Problem. When group $\mathcal{G}_n(w)$ is finite?

A low-dimensional topology point of views:

Problem. When group $\mathcal{G}_n(w)$ is fundamental group of a 3-manifold?

Example. The word $w(x_1, x_2, x_3) = x_1 x_2 x_3^{-1}$ leads to Fibonacci groups:

$$F(2, n) \ = \ \langle x_1, \ldots, x_n \quad | \quad x_i x_{i+1} \ = \ x_{i+2}, \quad i = 1, \ldots, n \rangle,$$

where subscripts are taken by mod n.

Example. The word $w(x_1, x_2, x_3) = x_1 x_3 x_2^{-1}$ leads to Sieradski groups:

$$S(n) \ = \ \langle x_1, \ldots, x_n \quad | \quad x_i x_{i+2} \ = \ x_{i+1}, \quad i = 1, \ldots, n \rangle,$$

where subscripts are taken by mod n

Remark. These groups are interesting from a topological point of views.

Fibonacci manifolds

Helling–Kim–Mennicke: Fibonacci groups $F(2, 2n)$, $n \geq 2$, arise as fundamental groups of closed orientable 3-manifolds M_n, called Fibonacci manifolds. For $n \geq 4$ manifolds M_n are hyperbolic.

Hilden–Lozano–Montesinos: Fibonacci manifolds M_n, $n \geq 2$, are n-fold cyclic coverings of the 3-sphere, branched over the figure-eight knot.

[2] H. Helling, A.C. Kim, J.L. Mennicke, A geometric study of Fibonacci groups. J. of Lie Theory 8 (1999), 1-23.

[3] H.M. Hilden, M.T. Lozano, J.M. Montesinos-Amilibia, The arithmeticity of the figure eight knot orbifolds. Topology'90, Columbus, 1990. Berlin: de Gruyter, 1992. 169-183.

Prop. [Johnson] If m is odd, then $F(2,m)$ has a torsion.

Proof. Indeed, consider

$$F(2,m) = \langle x_1, \ldots, x_m \quad | \quad x_i x_{i+1} = x_{i+2}, \quad i = 1, \ldots, m \rangle.$$

Taking $u = x_1 x_2 \ldots x_m$ we get

$$\begin{aligned} u^2 &= (x_1x_2)(x_3x_4)\cdots(x_mx_1)(x_2x_3)\cdots(x_{m-1}x_m) \\ &= x_3 \quad x_5 \quad \cdots \quad x_m \quad x_2 \quad x_4 \quad \cdots \quad x_1 \\ &= (x_2^{-1}x_4)(x_4^{-1}x_6)\cdots(x_{m-1}^{-1}x_1)(x_1^{-1}x_3)(x_3^{-1}x_5)\cdots(x_m^{-1}x_2) = 1. \end{aligned}$$

Now, let us verify that $u \neq 1$.

[4] D.L. Johnson, Topics in the theory of group presentations. London Math. Soc. Lect. Note Ser. 42, 1980.

Fibonacci groups with odd number of generators, II

The equality $u = 1$ will imply that all generators $x_1, \ldots, x_m$ commute.

Indeed, if $u = x_1 x_2 x_3 x_4 x_5 x_6 \cdots x_m = 1$, then

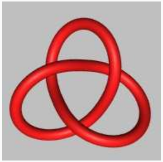
$$\begin{aligned}
x_3^2\, x_4\, x_5\, x_6\, \cdots\, x_m &= 1, \\
x_3\, x_5^2\, x_6\, \cdots\, x_m &= 1, \\
x_3\, x_5\, x_7^2\, \cdots\, x_m &= 1, \\
x_3\, x_5\, x_7\, \cdots\, x_m^2 &= 1, \\
(x_2^{-1} x_4)(x_4^{-1} x_6)(x_6^{-1} x_8) \cdots (x_{m-1}^{-1} x_1) x_m &= 1, \\
x_2^{-1}\, x_1\, x_m &= 1.
\end{aligned}$$

Comparing with $x_m x_1 = x_2$ we get that x_1 and x_m commute.

Analogously, x_i and x_{i+1} commute, and next, x_i and x_j commute.

But it is known that for odd $m \geq 9$ the group $F(2, m)$ is infinite and its abelianizer is finite. Hence, $u \neq 1$ and u is of order 2. □

Cavicchioli–Hegenbarth–Kim: Sieradski group $S(n)$ is fundamental group of the n-fold cyclic covering of S^3, branched over the trefoil knot.



Remark. Generalizations of Sieradski groups

$$S(k,n) = \langle x_1, \ldots, x_n \mid x_i x_{i+2} \ \ldots \ x_{i+2k-2} = x_{i+1} x_{i+3} \ \ldots \ x_{i+2k-3} \rangle$$

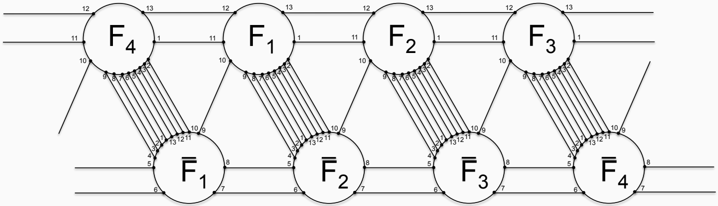
are fundamental groups of n-fold cyclic covering of the 3-sphere, branched over the torus knots $\mathbf{t}(2,k)$, where trefoil is $\mathbf{t}(2,3)$.

[5] A. Cavicchioli, F. Hegenbarth, A. Kim, A geometric study of Sieradski groups. Algebra Colloquium 5 (1998), 203-217.

An example of geometric cyclic presentation

Kozlovskaya – Vesnin: The following cyclic presentation is geometric:

$$\mathcal{G}_n(x_1 x_2 x_3 x_3 x_4 x_5 x_4^{-1} x_3^{-1} x_2 x_3 x_4 x_3^{-1} x_2^{-1}).$$



Heegaard diagram for the case $n = 4$.

The manifold is an n-fold cyclic branched cover of the lens space $L(5, 1)$.

[6]T. Kozlovskaya, A. Vesnin, Brieskorn manifolds, generalized Sieradski groups, and coverings of lens spaces, Proc. of IMM UrO RAN, 23 (2017), 85–97.

Defining words with 3 letters

Defining word with 3 letters

Cavicchioli – Hegenbarth – Repovs: are groups

$$G_n(m,k) = \langle x_1, x_2, \ldots, x_n \mid x_i x_{i+m} = x_{i+k}, \quad i = 1, \ldots, n \rangle.$$

fundamental groups of 3-manifolds?

If $(m,k) = (1,2)$ then we get Fibonacci groups

If $(m,k) = (2,1)$ then we get Sieradski groups.

Bardakov – Vesnin: If n is odd, $k - m$ is even, $(m - 2k, n) = 1$ then $G_n(m,k)$ cannot be fundamental group of a hyperbolic 3-manifold.

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[7] A. Cavicchioli, F. Hegenbarth, D. Repovs, On manifold spines and cyclic presentations of groups. Knot Theory, Banach Center Publications, 42 (1998), 49-56.

[8] V. Bardakov, A. Vesnin, On a generalisation of Fibonacci groups. Algebra and Logic, 42 (2003), 73-91.

Howie – Williams: Among $G_n(m, k)$ with the exception of two groups, only finite cyclic groups, Sieradski groups, and even-generated Fibonacci groups are fundamental groups of 3-manifolds.

Open Problem. Are groups

$$G_9(4,1) = \langle x_1, x_2, \ldots, x_9 \quad | \quad x_i x_{i+4} = x_{i+1}, \quad i = 1, 2, \ldots, 9 \rangle$$

and

$$G_9(7,1) = \langle x_1, x_2, \ldots, x_9 \quad | \quad x_i x_{i+7} = x_{i+1}, \quad i = 1, 2, \ldots, 9 \rangle$$

fundamental groups of 3-manifolds?

[9] E. Howie, G. Williams, Fibonacci type presentations and 3-manifolds. Topology Appl. 215 (2017), 24-34.

Last week results

Mohamed – Williams: Further study of groups $G_n(m,k)$. The orders of the abelianizations are calculated. Small cancelation properties and isomorphism classes of the groups with $n \leq 29$ and some infinite series are studded.

[10] E. Mohamed, G. Williams, An investigation into the cyclically presented groups with length three positive relations. arXiv:1806.06821, 18 June 2018.

Groups of 3-manifolds with cyclic symmetries

3-manifolds with cyclic symmetry, I

Assume that M is a closed orientable 3-manifolds which admit cyclic symmetries.

Birman and Hilden introduced a notion of n-symmetric Heegaard genus of M related to the symmetry of order n.

The *n-symmetric Heegaard genus* $g_n(M)$ of M is the smallest integer g such that M admits a n-symmetric Heegaard splitting of genus g.

Th. 1. [Birman – Hilden] Every closed orientable 3-manifold M of n-symmetric Heegaard genus $g_n(M)$ admits a representation as a n-fold cyclic covering of S^3 branched over a link L of bridge number

$$\mathrm{br}(L) \leq 1 + \frac{g_n(M)}{n-1}.$$

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[11] J.S. Birman, H.M. Hilden, Heegaard splittings fo branched coverings of S^3. Trans. AMS 213 (1975), 315-352.

Th. 2. [Birman – Hilden] The n-fold cyclic covering of S^3 branched over a knot of braid number b is a closed orientable 3-manifold M of n-symmetric Heegaard genus

$$g_n(M) \leq (b-1)(n-1).$$

Th. 3. [Mulazzani] A n-fold cyclic covering of S^3 branched over a knot K of bridge number $\mathrm{br}(K)$ is a closed, orientable 3-manifold M of n-symmetric Heegaard genus

$$g_n(M) \leq (\mathrm{br}(K)-1)(n-1).$$

Remark. We are interested in the case $\mathrm{br}(K) = 2$. Then $g_n(M) \leq n-1$.

[12] M. Mulazzani, On p-symmetric Heegaard splittings. JKTR 9 (2000), 1059-1067.

Rank and genus

For any n the n-symmetric Heegaard genus gives an upper bound for usual Heegaard genus of a 3-manifold M:

$$g(M) \leq g_n(M).$$

Denote by $\mathrm{rk}(G)$ the rank of a group G, i.e., the minimal number of its generators,

$$\mathrm{rk}(G) = \min\{|X| : \langle X \rangle = G\},$$

where $|X|$ is the cardinality of X.

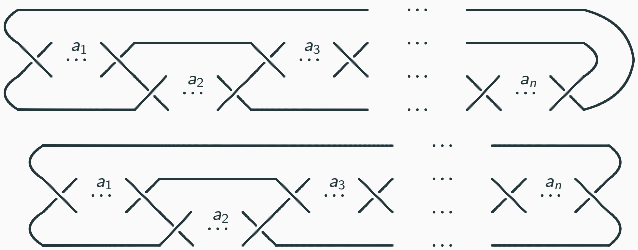
If G is fundamental group of closed orientable 3-manifold M, then

$$\mathrm{rk}(G) \leq g(M).$$

Two-bridge knots and links

The Conway normal form of a two-bridge link $\mathbf{b}(p/q)$.

$$p/q = [a_1, a_2, \ldots, a_{n-1}, a_n] = a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}.$$



Here a_j denotes a number of half-twists.

Denote by $M^{\varepsilon}_{k,\ell}(n)$, $\varepsilon = \pm 1$, an n-fold cyclic covering of S^3 branched over two-bridge knot $\mathbf{b}(p/q)$, where $p/q = 2k + \frac{1}{2\ell}$ and $p/q = 2k - \frac{1}{2\ell}$.

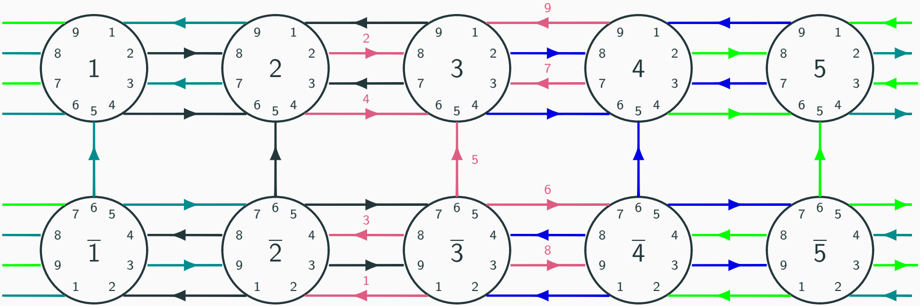
Th. 4. [Kim – Vesnin] The fundamental group $\pi_1(M^{\varepsilon}_{k,\ell}(n))$ has the following cyclic presentation

$$\pi_1(M^{\varepsilon}_{k,\ell}(n)) = \langle a_1, \ldots, a_n \,|\, (a_{i-1}^{-\ell} a_i^{\ell})^k a_i^{\varepsilon} (a_i^{-\ell} a_{i+1}^{\ell})^{-k} = 1, \quad i = 1, \ldots, n \rangle.$$

This presentation is geometric, i.e. it arises from the Heegaard splitting of the manifold. See picture for $\pi_1(M^{1}_{2,1}(5))$ on the next page.

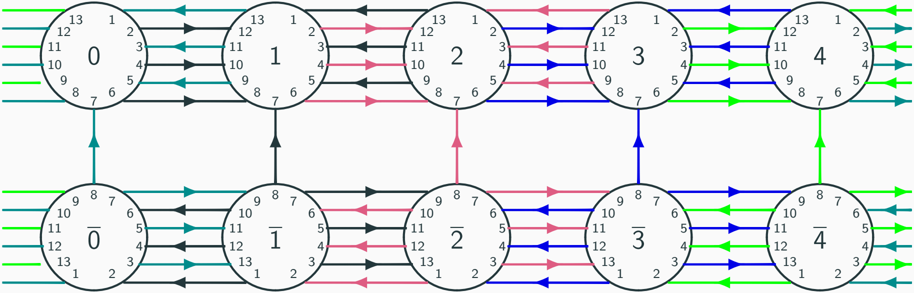
[13] A.C. Kim, A. Vesnin, Cyclically presented groups and Takahashi manifolds. RIMS Kokyuroku (Kyoto, Japan), 1022 (1998), 200-212.

Example of balanced presentation, I



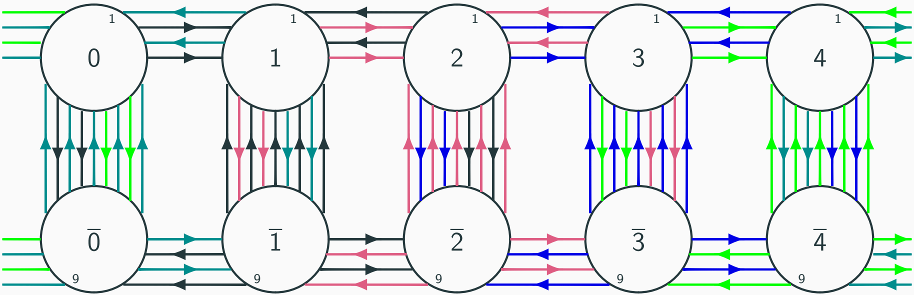
Heegaard diagram, corresponding to $\pi_1(M^1_{2,1}(5))$.

Example of balanced presentation, II



Heegaard diagram, corresponding to $\pi_1(M^1_{3,1}(5))$.

Example of balanced presentation, III



Heegaard diagram, corresponding to $\pi_1(M^1_{2,2}(5))$.

Th. 5. [Lei – Vesnin] Let $k, \ell \geq 1$, $\varepsilon = \pm 1$. Then for $n \geq 3$

$$g(M^{\varepsilon}_{k,\ell}(n)) \leq n - \left[\frac{n}{3}\right].$$

Method of the proof: destabilizaton moves for Heegaard diagrams.

Cor. The rank of the group $\pi_1(M^{\varepsilon}_{k,\ell}(n))$ is bounded as

$$\mathrm{rk}(\pi_1(M^{\varepsilon}_{k,\ell}(n))) \leq n - \left[\frac{n}{3}\right].$$

Open Problem. Find $\mathrm{rk}(\pi_1(M^{\varepsilon}_{k,\ell}(n)))$.

Remark. In almost all cases these manifolds are hyperbolic, hence $\mathrm{rk} \geq 2$.

[14] F. Lei, A. Vesnin, Work in progress.

Let $k = 1$, $\ell = 1$, $\varepsilon = 1$. Then we get $\mathbf{b}(5/2)$ that is the figure-eight knot. Then $\pi_1(M^1_{1,1}(n)))$ is the Fibonacci group $F(2, 2n)$ and it is two-generated.

Let $k = 2$, $\ell = 1$, $\varepsilon = -1$. Then we get $\mathbf{b}(7/2)$ that is the knot 5_2.

Th. 6. [Newman] The rank $\mathrm{rk}(\pi_1(M^{-1}_{2,1}(n)))$ satisfies the following inequalities:

$$\frac{\log 24}{\log 60}\left[\frac{n-6}{4}\right] \leq \mathrm{rk}(\pi_1(M^{-1}_{2,1}(n))) \leq \frac{n+1}{2}.$$

[15] M.F. Newman, On a family of cyclically presented fundamental groups, J. Austral. Math. Soc. 71 (2001), 235-241.

Motegi – Teragaito conjecture

A group G is said to be bi-orderable if G admits a strict total ordering $<$ which is invariant under the multiplication from left and right sides. That is, if $g < h$, then $agb < ahb$ for any $g, h, a, b \in G$. The trivial group $\{1\}$ is considered to be bi-orderable.

Let $g \in G$ be a non-trivial element. g is called a generalized torsion element if some non-empty finite product of conjugates of g equals to the identity.

Conjecture. [Motegi – Teragaito] Let G be the fundamental group of a 3-manifold. Then G is bi-orderable if and only if G has no generalized torsion element.

[16] K. Motegi, M. Teragaito, Generalized torsion elements and bi-oderability of 3-manifold groups. Canad. Mth. Bull. 60 (2017), 830-844.

Some known results, I

Fact. If G is bi-orderable, then G has no generalized torsion elements.

Proof. Let $<$ be bi-ordering of G. Suppose that G contains a generalized torsion element g. Therefore, there exist $a_1, \ldots, a_n \in G$ such that

$$g^{a_1} g^{a_2} \cdots g^{a_n} = 1,$$

where $g^a = a^{-1}ga$. Since $g \neq 1$, we have $g > 1$ or $g < 1$. If $g > 1$, then $g^{a_i} > 1$ for any i by bi-orderability. So, the product of these conjugates is still bigger than 1, a contradiction. $\square$

Some known results, II

Lemma. [Motegi – Teragaito] Let K be the Klein bottle. Then $\pi_1(K)$ contains a generalized torsion element.

Proof. It is well-known that

$$\pi_1(K) = \langle x, y \mid y^{-1}xy = x^{-1} \rangle.$$

Since $xx^y = 1$ from the relation and $x \neq 1$, x is a generalized torsion element. □

Th. 7. [Motegi – Teragaito] The fundamental group of any closed, geometric 3-manifold that is non-hyperbolic, satisfies Conjecture.

Th. 8. [Motegi – Teragaito] The fundamental group of the n-fold cyclic cover of S^3 branched over the figure-eight knot satisfies Conjecture.

Remark. That is the Fibonacci group $F(2, 2n)$.

Some known results, III

Prop. [Motegi – Teragaito] In the Fibonacci group $F(2, m)$, $m \geq 2$, each generator x_i is a generalized torsion element.

Method of the proof: combinatorial group theory.

Remark. This is the case $k = \ell = \varepsilon = 1$.

Problem. Which groups

$$\pi_1(M^{\varepsilon}_{k,\ell}(n)) = \langle a_1, \ldots, a_n \,|\, (a_{i-1}^{-\ell} a_i^{\ell})^k a_i^{\varepsilon} (a_i^{-\ell} a_{i+1}^{\ell})^{-k} = 1, \quad i = 1, \ldots, n \rangle$$

are bi-orderable?

Tran: Group $\pi_1(M_{2k,2\ell}(n))$ is left-oderable if

$$n > \pi / \cos^{-1} \sqrt{1 - (4k\ell)^{-1}}.$$

Dabkowski, Przytycki, Togha: Group $\pi_1(M^{-1}_{2k,2\ell}(n))$ for positive integers k and ℓ is not left-oderable for any integer $n > 1$.

[17] A. Tran, On left-oderability and cyclic branched coverings. J. Math. Soc. Japan 67 (2015), 1169-1178.

[18] M. Dabkowski, J. Przytycki and A. Togha, Non-left-orderable 3-manifold groups. Canad. Math. Bull. 48 (2005), 32-40.

Thank you!