Diamond Lemma and some applications

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- In 1942 M.H.A. Newman discovered a simple Lemma which turned to be useful in algebra and the theory of Grëbner basis.

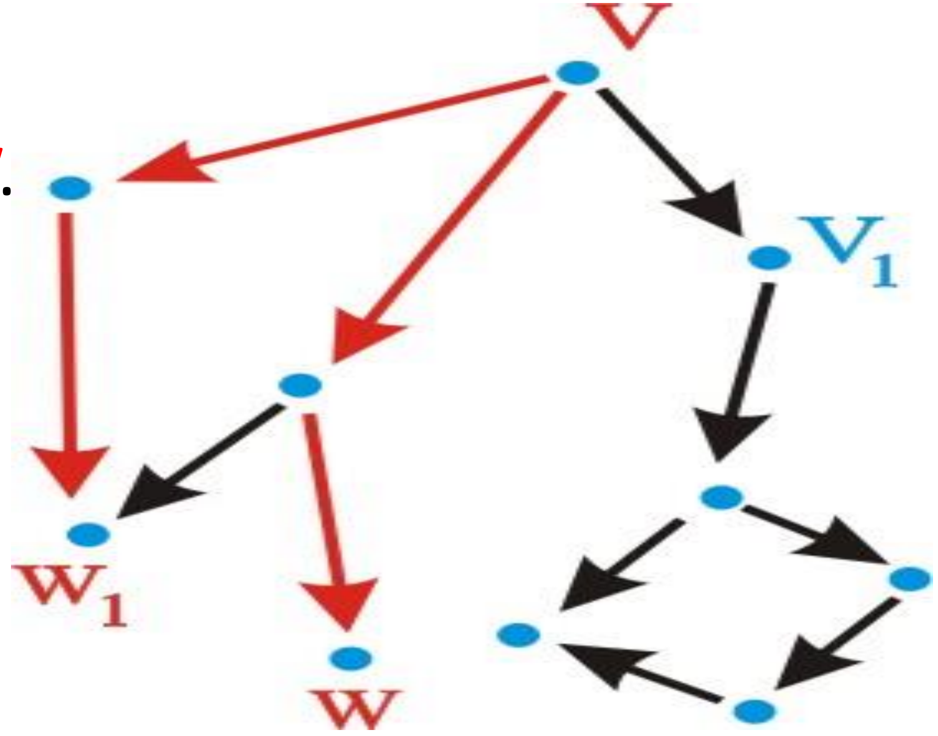
- In 2005 I introduced a new version of this Lemma and called it Diamond Lemma.
- It turned out that the new version is useful for working with topological objects: 3-manifolds, knots, knotted graphs.

Definitions

- Let Γ be an oriented graph.
- We denote its vertex and edge sets by $V(\Gamma)$ and $E(\Gamma)$.
- Oriented path in Γ is a sequence $\{\overrightarrow{A_1 A_2}, \overrightarrow{A_2 A_3} \ldots.\}$.

Definition 1. Vertex W is a root of V if

1. There is an oriented path from V to W.
2. Vertex W is a sink.



Question:

When the root of any vertex exists and is unique ?

We formulate two properties of graph Γ: (FP) and (MF).

- Finiteness property (FP) : *Any oriented path has finite length.*

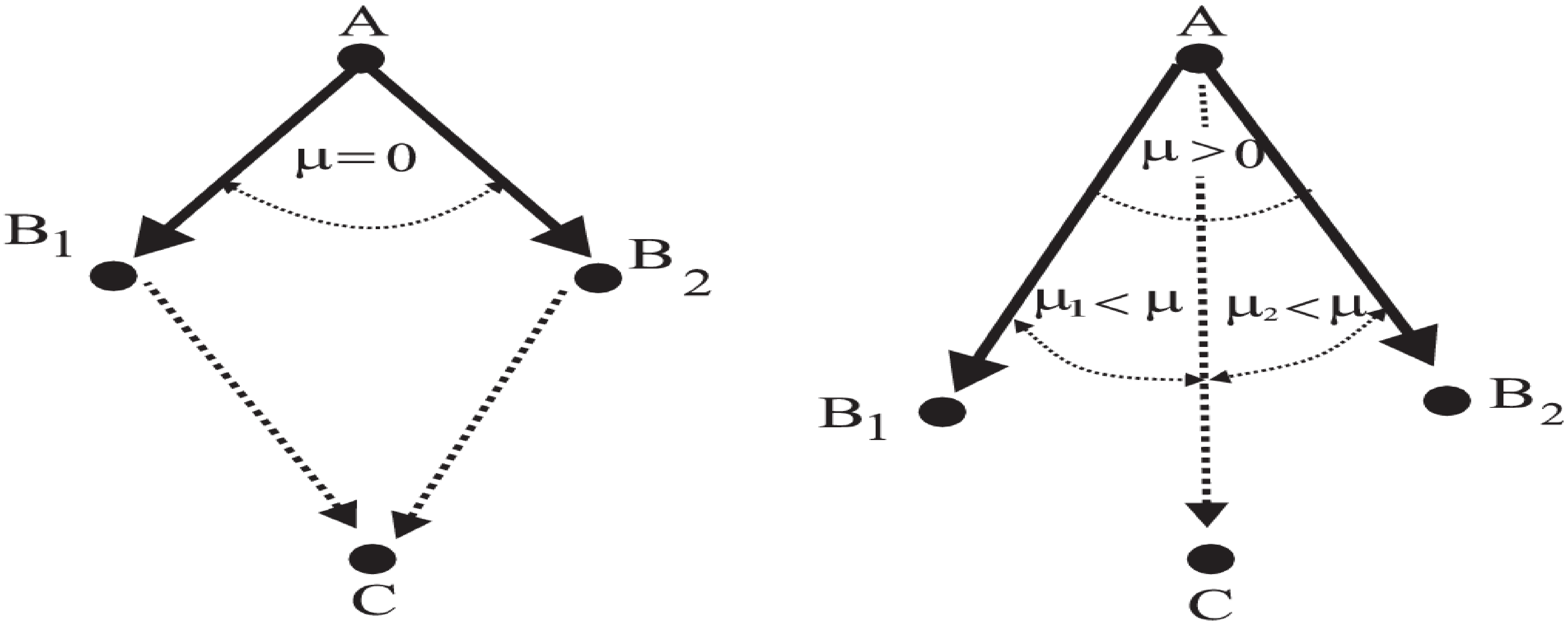
- It means that graph Γ does not contain oriented cycles or infinite oriented path.

- Denote $E^2(\Gamma)$ the set of all pair of edges in the form $(\overrightarrow{AB_1}, \overrightarrow{AB_2})$.

- Mediator function (MF): There exists the map

$\mu: E^2(\Gamma) \to \mathbb{N} \cup \{0\}$, called *mediator function,* which satisfies the following conditions:

- 1. For any edges $\left(\overrightarrow{AB_1}, \overrightarrow{AB_2}\right)$ with $\mu\left(\overrightarrow{AB_1}, \overrightarrow{AB_2}\right) = 0$ there exists a DIAMOND: vertex C and paths from B_1 and from B_2 ending in the same vertex C $\in V(\Gamma)$.
- 2. If $\mu\left(\overrightarrow{AB_1}, \overrightarrow{AB_2}\right) > 0$, then there exists the edge $\overrightarrow{AC}$ such that $\mu\left(\overrightarrow{AB_i}, \overrightarrow{AC}\right) < \mu\left(\overrightarrow{AB_1}, \overrightarrow{AB_2}\right)$ for $i = 1,2$.


A
μ=0
B1
B2
C
A
μ>0
μ1<μ
μ2<μ
B1
B2
C

Diamond Lemma

- *Suppose that an oriented graph Γ has property (FP) and (MF). Then every vertex has a unique root.*

- For proof we need same definitions and lemma.

Definitions

- 1. The vertex is called *regular* if it has exactly one root.
- 2. The vertex is called *singular* otherwise.
- 3. The edge is called *regular* if and only if its endpoint is regular.
- 4. The edge is called *singular* if and only if its endpoint is singular.

Lemma

Let an oriented graph have two properties:

- *1. (FP).*
- *2. graph Γ has a singular vertex.*

Then there exists singular vertex with all regular outgoing edges.

- Proof is easy: arguing by contradiction, assume that any singular vertex of graph Γ has singular outgoing edge. Choose the path of these singular outgoing edges. We get infinite path in contradiction to (FP).

Proof of Diamond Lemma

- Arguing by contradiction, assume that the graph Γ has the singular vertex A with outgoing regular edges (according to the last lemma).

- Choose two edges $\overrightarrow{AB_1}$ and $\overrightarrow{AB_2}$ which have different roots, and the number $\mu_0 = \mu(\overrightarrow{AB_1}, \overrightarrow{AB_2})$ is minimal possible among all numbers $\mu\left(\overrightarrow{AX_1}, \overrightarrow{AX_2}\right)$ with X_1 and X_2 having different roots. Note that $B_1 \neq B_2$ since both B_1 and B_2 are regular.

- We have two possibilities: $\mu\left(\overrightarrow{AB_1}, \overrightarrow{AB_2}\right) = 0$ and $\mu\left(\overrightarrow{AB_1}, \overrightarrow{AB_2}\right) > 0.$
- In the first case any root for vertex *C* in (MF1) is a common root for vertices B_1 and B_2. Since both these vertices are regular, they do not have other roots, which contradicts the choice of B_1 and B_2.
- In the second case we obtain a contradiction between the property (MF2) and the choice of the edges $\overrightarrow{AB_1}$ and $\overrightarrow{AB_2}$.

Let vertex *C* be the end of edge $\overrightarrow{AC}$ in (MF2). All the vertices B_1, B_2 and *C* are regular, each of them has unique root.

- The roots are different, and therefore at least one of these roots (let say root of B_2) is different from root of *C*.
- Since according to (MF2)

$$\mu\left(\overrightarrow{AB_2}, \overrightarrow{AC}\right) < \mu\left(\overrightarrow{AB_1}, \overrightarrow{AB_2}\right) = \mu_0.$$

- This is the contradiction.

The universal application scheme

- Let $\mathbb{Q}$ be the class of objects under consideration.
- We first construct a graph Γ whose vertices are unordered finite sets of objects in the class $\mathbb{Q}$.
- The edges of the graph are defined by means of non-trivial reductions.

- Reductions of objects define reductions of vertices of the graph, and therefore define its edges. Two vertices A, B of Γ are joined by an edge $\overrightarrow{AB}$ if and only if they consist of the same objects with one exception: one object at the vertex A undergoes a reduction, that is, is replaced by a new object or pair of objects.

- An edge is determined by the initial and terminal sets of objects only, and does not depend on the particular way in which a reduction is carried out. Hence, the graph Γ does not contain multiple edges.

Properties (FP) and (MF)

- The proof of the property (FP) is usually straightforward. It depends on the particular problem.
- By contrast, the mediator function in (MF) is always constructed in the same way.

- Its value $\mu(\overrightarrow{AB_1}, \overrightarrow{AB_2})$ at a pair of edges is defined as the minimal number $\#(S_1 \cap S_2)$ of connected components of the intersection of surfaces $S_1,\ S_2 \subset A$ defining the edges $\overrightarrow{AB_1},\ \overrightarrow{AB_2}$, where the minimum is taken over all pairs of such surfaces. Of course, here we assume that the surfaces are in general position.

- If $\mu(\overrightarrow{AB_1}, \overrightarrow{AB_2}) = 0$, that is, if the edges can be defined by disjoint surfaces, then each of these surfaces survives under reduction along the other surface. This implies that the vertex obtained as a result of both reductions does not depend on which of the reductions is carried out first. Therefore, it can be taken as the vertex C in the formulation of (MF1).

- Assume now that $\mu(\overrightarrow{AB_1}, \overrightarrow{AB_2}) > 0$. This implies that the vertices $B_1, \ B_2$ are different, since otherwise the edges $\overrightarrow{AB_1}$ and $\overrightarrow{AB_2}$ could be given by parallel copies of the same surface, which would imply that $\mu\left(\overrightarrow{AB_1}, \overrightarrow{AB_2}\right) = 0$. In order to establish the property (MF2) it is enough to show that for any two surfaces $S_1, S_2 \subset A$ defining the edges $\overrightarrow{AB_1}, \overrightarrow{AB_2}$ there exists a surface $S \subset A$ defining an edge and satisfying $\sharp(S_i \cap S) < \sharp(S_1 \cap S_2)$ for $i = 1,2$.

- We refer to this S as a mediator surface. The remarkable fact is that the standard technique of removing intersections of surfaces can be applied to find a mediator surface.
- This technique has long been used in low-dimensional topology and is quite well developed. Of course, the particular methods for constructing mediator surfaces depend on the nature of the problem.

Applications:

- 1. The Kneser-Milnor prime decomposition theorem:

A compact connected orientable 3-dimensional manifold M different from a sphere is gomeomorphic to a connected sum $M_1 \# M_2 \# \ldots \# M_n \# m(S^1 \times S^2)$, *where all the* M_i *are irreducible and* $m(S^1 \times S^2)$ *denotes connected sum of* $m \geq 0$ *copies of* $S^1 \times S^2$. *All summands are defined uniquely by reordering.*

For proof we could using reductions along all spheres.

For nonorientable 3-manifolds this theorem is false!

The theorem is true by replacing

$$S^1 \times S^2 \longleftrightarrow S^1 \tilde{\times} S^2$$

The other results

2. The Swarup theorem for boundary connected sums (new proof).

3. A spherical splitting theorem for knotted graphs in 3-manifolds (Joint work with C. Hog-Angeloni; After Petronio's splittings of 3-orbifolds) (new proof).

- 4. Counterexamples to prime decomposition theorems for knots in 3-manifolds and for 3-orbifolds.

- 5. A new theorem on annular splittings of 3-manifolds, which is independent of the JSJ-decomposition theorem.

6. An existence and uniqueness theorem for prime decompositions of homologically trivial knots in direct products of surfaces and intervals.

7. A theorem on the exact structure of the semigroup of theta-curves in 3-manifolds

(joint work with V. Turaev).

Schubert prime decomposition theorem:

(1) Any knot in the 3-sphere can be decomposed into connected sum of prime factors.

(2) The factors are unique up to permutation.

Does the Schubert Theorem hold for global knots?

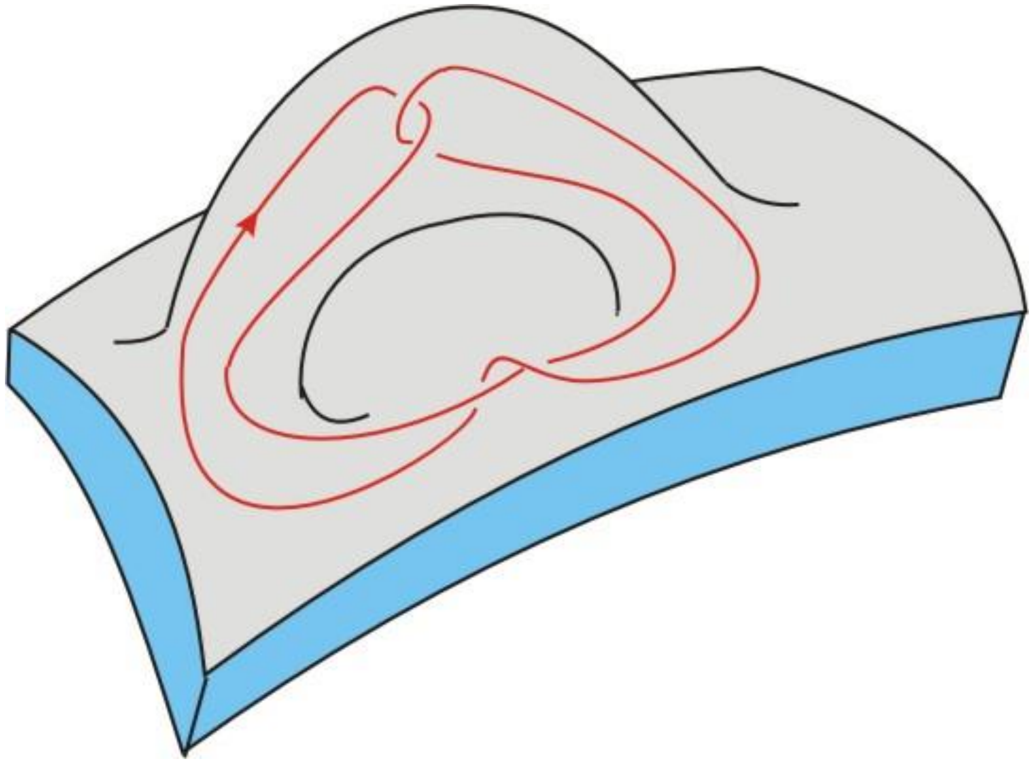
- Existence: in general NO!

If a knot K in a 3-manifold M crosses
a 2-sphere in M at exactly one point
then K has no prime decomposition.

- If there is no such sphere then prime decompositions of K do exist.

Knots in direct products of surfaces by interval

- F x I are simplest 3-manifolds after S^3
- Knots in F x I have classical diagrams
- They dominate virtual knots



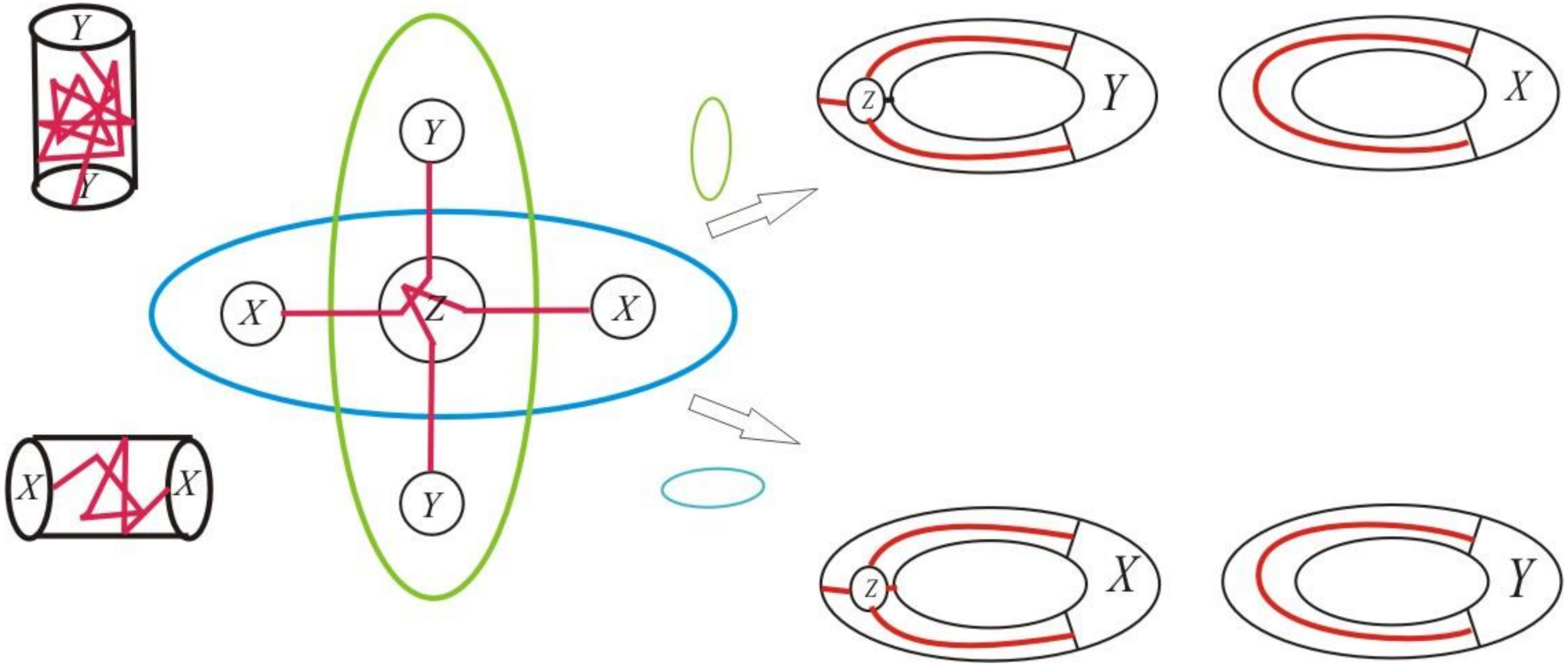
Questions:

- Does any knot have a prime decomposition? Yes!
- Are the summands unique? Yes and No!
- Main Theorem:

If $[K] \in H_1(F; Z_2)$ is 0 then Yes

In general No

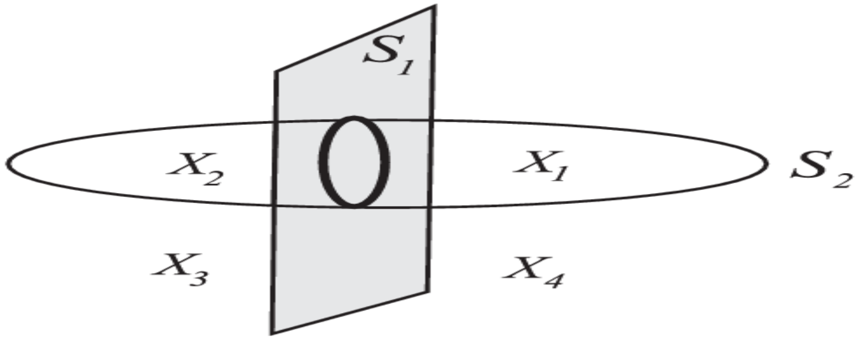
Counterexample

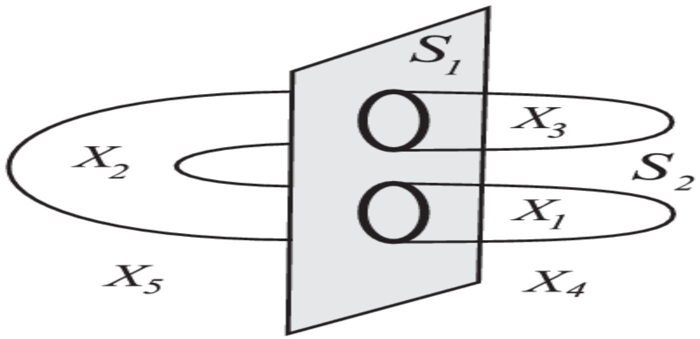


Thank you for your attention

- The end

A mediator sphere can be found among the spheres ∂X_i (left) or among the spheres ∂X_i and the sphere $\partial X_1 \# \partial X_3$ (right)

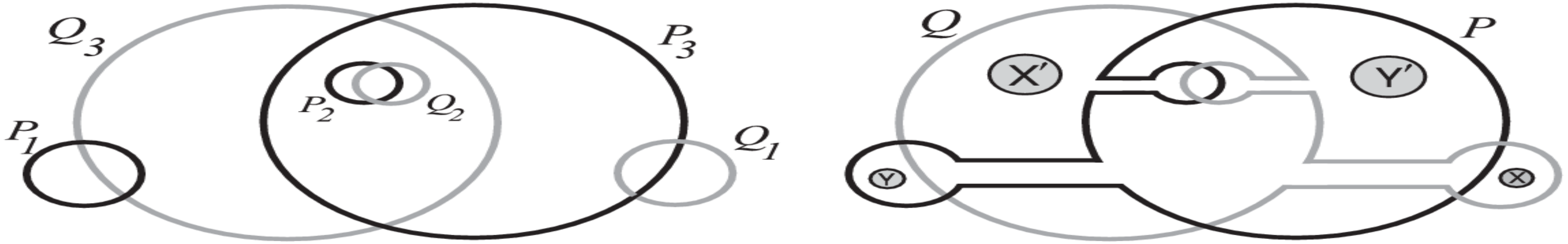




2. A spherical splitting theorem for knotted graphs in 3-manifolds (Joint work with C. Hog-Angeloni; After Petronio's splittings of 3-orbifolds)

- *Let (M,G) be a knotted graph, and apply non-trivial spherical reductions to it as many times as possible. Then this process always stops after a finite numbers of steps. The resulting set of irreducible knotted graphs is defined uniquely up to addition or deletion of trivial pairs in form of a suspension (double cone) over the pair* (S^2, X)*, where the set* $X \subset S^2$ *consists of* $k \leq 3$ *points.*

Two splitting spheres in manifold $(S^1 \times S^2)\#(S^1 \times S^2)$



Counterexample for knotted graphs: along the sphere *P* and along the sphere *Q*