Residual finiteness of quandles

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Overview

Quandles

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Definition

A *quandle* is a non-empty set X with a binary operation $(x, y) \mapsto x * y$ satisfying the following axioms:

1. $x * x = x$ for all $x \in X$;
2. For any $x, y \in X$ there exists a unique $z \in X$ such that $x = z * y$;
3. $(x * y) * z = (x * z) * (y * z)$ for all $x, y, z \in X$.

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- *Coset quandle* (G, H, z): Let $z \in G$ and H be a subgroup of G contained in $C_G(z)$. Then G/H is a quandle as

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- Let $\{z_i \mid i \in I\}$ elements of G, and $\{H_i \mid i \in I\}$ subgroups of G and H_i is contained in the centralizer $C_G(z_i)$ for each $i \in I$. Then

$$Q = \sqcup_{i \in I}(G, H_i, z_i)$$

is a quandle with

$$H_ix * H_jy = H_iz_i^{-1}xy^{-1}z_jy.$$

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Two links L_1 and L_2 are said to be isotopic if there exist an isotopic deformation $\{h_t\}$ of $\mathbb{S}^3$ such that $h_1(L_1) = L_2$.

Relation between quandle and knot theory

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In 1982 Matveev and Joyce (independently) associated to each oriented knot K a quandle $Q(K)$ called the *knot quandle* and proved that the knot quandle is "almost" a complete invariant. Since then quandles have been investigated in order to construct knot and link invariants.

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Construction of a link quandle

- Let L be an oriented link in $\mathbb{S}^3$ with components $K_1, K_2, \dots, K_n$.
- $V(L) :=$ Tubular neighborhood of L.
  $V(L) = V(K_1) \sqcup V(K_2) \sqcup \dots \sqcup V(K_n)$, where $V(K_i)$ is a tubular neighborhood of K_i.
- Link complement $C(L) := \overline{\mathbb{S}^3 - V(L)}$.
- Fix a base point $x_0 \in C(L)$.

Relation between quandle and knot theory

Construction of a link quandle

- $Q(L) :=$ Set of homotopy classes of paths in $C(L)$ with initial point on the boundary $\partial\big(V(L)\big)$ and end point at x_0.
- Let $[a]$ and $[b] \in Q(L)$;
- $[a] * [b] := [ab^{-1}m_{b(0)}b]$.

Then $Q(L)$ is a quandle associated to link L and is known as the *link quandle*.

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Theorem (Matveev-Joyce)

Let K and K' be two oriented knots in the $\mathbb{S}^3$. Then, K is isotopic to either K or $-K'^*$ if and only if there exists a quandle isomorphism between the knot quandles Q_K and Q'_K.

Residual finiteness property

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Definition

A group G is said to be *residually finite* if for all $g, h \in G$ with $g \neq h$, there exists a finite group F and group homomorphism $\phi : G \to F$ such that $\phi(g) \neq \phi(h)$.

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Residual finiteness property and knot theory

Residual finiteness property and knot theory

- Neuwirth (1965) showed that knot groups of fibered knots are residually finite and conjectured that every knot group is residually finite.
- Mayland (1972) proved it of twist knots, and Stebe extended the result to certain class of non-fibered knots.
- Thurston (1982) proved that knot groups are residually finite.
- Perelman's proof of the geometrization conjecture implies that the fundamental group of every compact 3-manifold is resdiually finite.

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Every finitely generated residually finite quandle is Hopfian.

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Definition

Let X be a quandle. For each $x \in X$ the map $S_x : X \to X$ defined as $S_x(y) := y * x$ is called an inner automorphism. The group generated by all such automorphisms is called *inner automorphism group of quandle X and denoted by* $\mathrm{Inn}(X)$.

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Inner automorphism group of a residually finite quandle is residually finite.

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Theorem (Belk-McGrail)

There exists a finitely presented quandle with undecidable word problem.

Theorem

Every finitely presented residually finite quandle has a solvable word problem.

First Main Result

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Theorem

Knot quandles are residually finite.

V.G. Bardakov, M. Singh and M. Singh, *Free quandles and knot quandles are residually finite*, Proc. Amer. Math. Soc.(2019)

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Theorem

Let G be a group and H a subgroup such that $H \leq C_G(z)$. If H is finitely separable in G, then (G, H, z) is a residually finite quandle.

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Theorem (Long-Niblo, 1991)

Suppose that M is an orientable, irreducible compact 3-manifold and X an incompressible connected subsurface of a component of $\partial(M)$. If $p \in X$ is a base point, then $\pi_1(X, p)$ is a finitely separable subgroup of $\pi_1(M, p)$.

Second Main Result

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Theorem

Link quandles are residually finite.

V.G. Bardakov, M. Singh and M. Singh, *Link quandles are residually finite*, arXiv:1902.03082

Outline of the proof

Main steps:

1. Knot quandles are residually finite.
2. Link quandle corresponding to non-split links are residually finite.(Proof follows same approach as in case knots.)
3. Let $L = \{L_1, L_2, \ldots, L_k\}$ where L_i are non split links. Then $Q(L) = Q(L_1) * Q(L_2) * \cdots * Q(L_k)$.
4. Examine the free product of residually finite quandles.

Free product of quandles

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Definition

Let $A = \langle X \mid R \rangle$ and $B = \langle Y \mid S \rangle$ be two quandles with non-intersecting set of generators. The free product $A * B$ is a quandle defined by the presentation

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Definition

The *associated group* $\mathrm{As}(Q)$ of a quandle Q is defined to be the group generated by the set $\{e_x \mid x \in Q\}$ modulo the relations $e_{x*y} = e_y^{-1} e_x e_y$ for all $x, y \in Q$.

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Theorem

Let $\{Q_i\}_{i \in I}$ be a family of residually finite quandles. If each $\mathrm{As}(Q_i)$ is a residually finite group, then the free product $\star_{i \in I} Q_i$ is a residually finite quandle.

Outline of the proof

Theorem (Matveev-Joyce)

For any link L, the associated group $\mathrm{As}\big(Q(L)\big)$ is isomorphic to link group $G(L)$.

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1. Knot quandles are residually finite.
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3. Let $L = \{L_1, L_2, \dots, L_k\}$ where L_i are non split links. Then $Q(L) = Q(L_1) * Q(L_2) * \cdots * Q(L_k)$.
4. The free product of residually finite quandles is residually finite provided their associated groups are residually finite.
5. Link quandles are residually finite.

Corollary

Let L be a link in $\mathbb{S}^3$. Then the following are true:

1. *Word problem is solvable in $Q(L)$.*
2. $\mathrm{Inn}(Q(L))$ *is residually finite.*
3. $Q(L)$ *is Hopfian.*

Problems

1. Let X be a finitely generated residually finite quandle. Is it true that $\text{Aut}(X)$ is a residually finite group?
2. Let X be a residually finite quandle. Is it true that $\text{As}(X)$ is residually finite?
3. Is it true that free product of residually finite quandles is residually finite?

Thank you!