Knots, Quandles and Invariants

Groups and quandles in low-dimensional topology

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- A knot K is an embedding of a circle $\mathbb{S}^1$ into the 3-sphere $\mathbb{S}^3$.
- Two knots K_1 and K_2 are said to be equivalent if K_1 can be transformed into K_2 via an ambient isotopy.
- A knot is called tame if it is equivalent to a polygonal knot (or a smooth knot).
- We consider only tame and oriented knots.

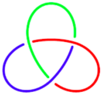
A Basic Problem

Given two knots K_1 and K_2, are they equivalent?

- A knot invariant is a function that assigns a quantity or a mathematical expression to each knot which is preserved under the knot equivalence.
- Examples of knot invariants:
    - Unknotting Number = The minimal number of crossing switches needed to unknot a knot.
    - Knot group $G(K) := \pi_1(\mathbb{S}^3 \setminus K)$.
    - 3-Coloring.
    - Alexander polynomial.
    - Jones polynomial.
    - Kauffman polynomial.
    - Quandle homology.

    $\vdots$
- None of the above is a complete invariant.
- However, $G(K) \cong \mathbb{Z}$ if and only if K is a trivial knot [Papakyriakopoulos, 1957].

- An elementary knot invariant is the number of 3-colorings.
- A 3-coloring of a knot diagram $D(K)$ is an assignment to each arc one of the three colors (say, red, blue, green) such that any three incident arcs are either all the same color or all different colors.
- Here is a non-trivial 3-coloring of the trefoil knot.



Theorem

Any two knot diagrams related by Reidemeister moves have the same number of 3-colourings.

- The total number of 3-colorings is a knot invariant, denoted $Col_3(K)$.

- A quandle is a set X with a binary operation $(a, b) \mapsto a * b$ satisfying the following conditions:
  1. $x * x = x$ for all $x \in X$;
  2. For any $x, y \in X$, there is a unique $z \in X$ such that $x = z * y$;
  3. $(x * y) * z = (x * z) * (y * z)$ for all $x, y, z \in X$.
- Introduced independently by Matveev and Joyce in 1982.
- Equivalently, for each element $x \in X$, the map $S_x : X \to X$ given by

$$S_x(y) = y * x$$

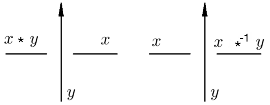
  is an automorphism of X fixing x, referred as inner automorphism.
- S_x being a bijection is equivalent to existence of another binary operation on X, $(x, y) \mapsto x *^{-1} y$, satisfying

$$x * y = z \text{ if and only if } x = z *^{-1} y$$

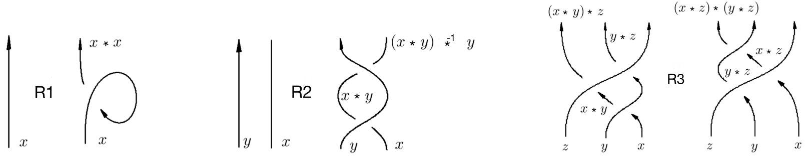
  for all $x, y, z \in X$.

Quandle axioms vs Reidemeister moves

- Quandle axioms are derived from the Reidemeister moves on oriented knot diagrams.
- For each crossing of a knot diagram, we set



- The three quandle axioms are equivalent to the three Reidemeister moves of knot diagrams.



- Many interesting examples of quandles come from groups.
  1. If G is a group, then the set G with the binary operation
  $$a * b = b^{-1}ab$$
  gives a quandle structure on G, called conjugation quandle, and denoted by $\mathrm{Conj}(G)$.
  2. Let G be a group and $\varphi \in \mathrm{Aut}(G)$. Then the set G with binary operation
  $$a * b = \varphi(ab^{-1})b$$
  gives a quandle structure on G, referred as generalized Alexander quandle, and denoted by $\mathrm{Alex}(G, \varphi)$.
  If G is additive abelian and $\varphi = -\mathrm{id}_G$, then $a * b = 2b - a$, and the quandle is called Takasaki quandle.
  In addition, if $G = \mathbb{Z}/n\mathbb{Z}$, then it is called dihedral quandle, and denoted by R_n.
  3. A Riemannian manifold M is called a symmetric space if for each $x \in M$ there exists a globally defined symmetry $S_x : M \to M$. Every symmetric space is a quandle with the binary operation given by $y * x = S_x(y)$.

- If X is a quandle, then its *enveloping group* $\mathrm{Env}(X)$ is defined by generators as elements of X and relations given by

$$x * y = y^{-1}xy$$

for $x, y \in X$.

- Env is a functor from the category of quandles to that of groups.
- Further, Conj is also a functor from the category of group to that of quandles.

Proposition [Matveev/Joyce, 1982]

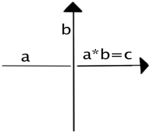
The functor Env is the left adjoint to the functor Conj. Namely, for a quandle X and a group G, there is a natural bijection

$$\mathrm{Hom}_{Groups}(\mathrm{Env}(X), G) \cong \mathrm{Hom}_{Quandles}(X, \mathrm{Conj}(G)).$$

- As expected, knots give rise to quandles.
- If K is a knot, then the knot quandle is defined as

$$Q(K) := \langle Arcs\ in\ D(K) \mid \mathcal{R} \rangle,$$

where the set of relations $\mathcal{R}$ consists of expressions $a * b = c$ whenever the arc b passes over the double point separating arcs a and c.



Theorem [Matveev/Joyce, 1982]

Let K_1 and K_2 be two knots. Then K_1 is equivalent to K_2 (up to orientation) if and only if $Q(K_1) \cong Q(K_2)$.

- Unfortunately, it is very hard to work with freely presented quandles.

- 3-coloring of knots (links) was generalized by Fox.
- An n-coloring of a knot (link) diagram $D(K)$ is an assignment to each arc one of the numbers $\{0, 1, \ldots, n-1\}$ (called colors) such that at each crossing the sum of the colors of the undercrossings is equal to twice the color of the overcrossing modulo n.

Theorem [Fox, 1962]

Reidemeister moves preserve the number of n-colorings.

- Hence the number of n-colorings $Col_n(K)$ is a knot invariant.
- Viewing the set $\{0, 1, \ldots, n-1\}$ as the Dihedral quandle R_n, the number $Col_n(K)$ is simply the number of quandle homomorphisms from the knot quandle $Q(K)$ to R_n.

- Fox's idea can be extended to arbitrary quandles.
- Given a knot K and a quandle X, a quandle coloring of K by X is a quandle homomorphism from $Q(K)$ to X.

Theorem

Given a quandle X, the number of quandle colorings $|\mathrm{Hom}(Q(K), X)|$ is a knot invariant.

- The dihedral quandle R_3 corresponds to $Col_3(K)$.
- In general, R_n corresponds to Fox's n-coloring $Col_n(K)$.

- An action of a quandle Q on a quandle X is a quandle homomorphism
$$\phi : Q \to \mathrm{Conj}\big(\mathrm{Aut}(X)\big),$$
where $\mathrm{Aut}(X)$ is the group of all quandle automorphisms of X, and $\mathrm{Conj}\big(\mathrm{Aut}(X)\big)$ its conjugation quandle.
- Action is trivial if $\mathrm{Im}(\phi) = \{\mathrm{id}_X\}$.
- Notice that, any set X can be viewed as a trivial quandle. In that case, $\mathrm{Aut}(X) = \Sigma_X$, the group of all bijections of the set X, and we obtain the definition of an action of a quandle Q on a set X.

- If Q is a quandle, then the map $\phi : Q \to \text{Conj}\big(\text{Aut}(Q)\big)$ given by $q \mapsto S_q$ is a quandle homomorphism. Thus, every quandle acts on itself by inner automorphisms.
- Let G be a group acting on a set X. That is, there is a group homomorphism $\phi : G \to \Sigma_X$. Viewing both G and Σ_X as conjugation quandles and observing that a group homomorphim is also a quandle homomorphism, it follows that the quandle $\text{Conj}(G)$ acts on the set X.

- Let Q and X be two quandles and $\phi : Q \to \mathrm{Conj}\big(\mathrm{Aut}(X)\big)$ a quandle action of Q on X. A map $f : Q \to X$ satisfying

$$f(q_1 * q_2) = f(q_1) * f(q_2)^{\phi(q_1)}$$

for all $q_1, q_2 \in Q$, is called a derivation with respect to the quandle action ϕ of Q on X.

- $\mathrm{Der}_\phi(Q,X) := \{f : Q \to X \mid f \text{ is a derivation with respect to } \phi\}$. If the action ϕ is trivial, then $\mathrm{Der}_\phi(Q,X) = \mathrm{Hom}(Q,X)$, the set of all quandle homomorphisms from Q to X.
- Given a non-trivial action ϕ of a quandle Q on a non-trivial quandle X, it is possible that the set $\mathrm{Der}_\phi(Q,X)$ is empty.
- However, we can always find non-trivial actions of Q on X for which this set is non-empty. Let $\mathrm{id}_X \neq S_x \in \mathrm{Inn}(X)$ and $\phi : Q \to \mathrm{Conj}\big(\mathrm{Aut}(X)\big)$ given by

$$\phi(q) = S_x$$

for $q \in Q$. Then $f : Q \to X$ defined as

$$f(q) = x$$

for $q \in Q$, is clearly an element of $\mathrm{Der}_\phi(Q,X)$.

- A quandle X is said to be abelian/medial if

$$(x * y) * (z * w) = (x * z) * (y * w)$$

for all $x, y, z, w \in X$.
- For example, if A is an additive abelian group, then the Takasaki quandle $T(A)$ is abelian.
- A quandle X is said to be commutative if

$$x * y = y * x$$

for all $x, y \in X$.
- Unlike groups, being commutative and being abelian do not mean the same for quandles. In fact, any trivial quandle with more than one element is abelian but not commutative. The dihedral quandle R_3 on three elements is both abelian and commutative.

Theorem [NSS, 2018]

Let Q and A be quandles such that A is abelian and $\phi : Q \to \mathrm{Conj}(\mathrm{Aut}(A))$ a quandle action. If the set $\mathrm{Der}_\phi(Q, A)$ is non-empty, then it has the structure of an abelian quandle with respect to the binary operation

$$(f * g)(q) = f(q) * g(q)$$

for $f, g \in \mathrm{Der}_\phi(Q, A)$ and $q \in Q$.

Compatible quandle actions

- Let Q_1, Q_2 be two quandles and A_1, A_2 two abelian quandles. Let

$$\phi_1 : Q_1 \to \mathrm{Conj}\left(\mathrm{Aut}(A_1)\right)$$

and

$$\phi_2 : Q_2 \to \mathrm{Conj}\left(\mathrm{Aut}(A_2)\right)$$

be given actions.

- A pair of quandle homomorphisms $\sigma : Q_2 \to Q_1$ and $\tau : A_1 \to A_2$ is said to be action compatible if the following diagram commutes

$$\begin{array}{ccc} A_2 \times Q_2 & \xrightarrow{\widetilde{\phi_2}} & A_2 \\ {\scriptstyle\tau}\uparrow \quad \downarrow{\scriptstyle\sigma} & & \uparrow{\scriptstyle\tau} \\ A_1 \times Q_1 & \xrightarrow{\widetilde{\phi_1}} & A_1. \end{array}$$

Theorem-I [NSS, 2018]

Let Q_1, Q_2 be two quandles and A_1, A_2 two abelian quandles. Let $\phi_1 : Q_1 \to \text{Conj}\left(\text{Aut}(A_1)\right)$ and $\phi_2 : Q_2 \to \text{Conj}\left(\text{Aut}(A_2)\right)$ be actions of Q_1, Q_2 on A_1, A_2, respectively. Let $\sigma : Q_2 \to Q_1$ and $\tau : A_1 \to A_2$ be action compatible quandle homomorphisms. Then there exists a quandle homomorphism

$$\Phi : \text{Der}_{\phi_1}(Q_1, A_1) \to \text{Der}_{\phi_2}(Q_2, A_2).$$

Further, if σ and τ are both isomorphisms, then so is Φ.
Additionally, if Q_1, Q_2 are finitely generated and A_1, A_2 are finite, then

$$|\,\text{Der}_{\phi_1}(Q_1, A_1)| = |\,\text{Der}_{\phi_2}(Q_2, A_2)|.$$

Theorem [NSS, 2018]

Derivation quandles of a tame knot with respect to an abelian quandle are knot invariants.

Proof: Let K_1 and K_2 be two equivalent tame knots with knot quandles $Q(K_1)$ and $Q(K_2)$, respectively. Then, by Joyce/Matveev, there is an isomorphism $\sigma : Q(K_2) \to Q(K_1)$. Let A be an abelian quandle and $\phi_1 : Q(K_1) \to \mathrm{Conj}\big(\mathrm{Aut}(A)\big)$ an action of $Q(K_1)$ on A. Then $\phi_2 := \phi_1 \circ \sigma$ is an action of $Q(K_2)$ on A. By Theorem-I, we obtain an isomorphism $\mathrm{Der}_{\phi_1}\big(Q(K_1), A\big) \cong \mathrm{Der}_{\phi_2}\big(Q(K_2), A\big)$. Thus, derivation quandles are knot invariants.

- Given two quandles $(X_1, *_1)$ and $(X_2, *_2)$, the disjoint union $X_1 \sqcup X_2$ can be turned into a quandle by defining

$$x * y = \begin{cases} x *_1 y & \text{if } x, y \in X_1 \\ x *_2 y & \text{if } x, y \in X_2 \\ x & \text{if } x \in X_1, y \in X_2 \\ x & \text{if } x \in X_2, y \in X_1. \end{cases} \tag{1}$$

- If X_1 and X_2 are abelian, then $X_1 \sqcup X_2$ is not abelian in general.
- Let K be a tame knot and A an abelian quandle. Taking $X_1 = \mathrm{Hom}\big(Q(K), A\big)$ and

$$X_2 = \bigsqcup_{\phi \text{ non-trivial action}} \mathrm{Der}_{\phi}\big(Q(K), A\big),$$

we get a non-abelian quandle

$$\mathcal{D}\big(Q(K), A\big) := X_1 \sqcup X_2,$$

called the total derivation quandle.

Theorem

The total derivation quandle with respect to an abelian quandle is an invariant of tame knots, and contains the hom quandle as an abelian subquandle.

The knots

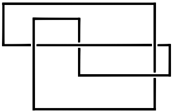


Figure : Figure Eight Knot 4_1

and

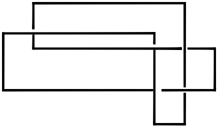


Figure : Knot 5_2

have isomorphic hom quandles, but total derivation quandles are non-isomorphic, in fact, of different sizes.

- Let Q be a quandle and X a finite quandle (not necessarily abelian). The derivation polynomial of Q with respect to X is defined as

$$D_X(Q)(u) = \left|\operatorname{Hom}\left(Q, X\right)\right| + \sum_{\phi \text{ non-trivial action}} u^{\left|\operatorname{Der}_\phi(Q,X)\right|+1}.$$

Theorem [NSS, 2018]

The derivation polynomial of a tame knot with respect to a finite quandle is a knot invariant.

- Let K be a tame knot, X a finite quandle and $D_X(K)(u)$ the derivation polynomial. Then $D_X(K)(0) = \left|\operatorname{Hom}\left(Q(K), X\right)\right|$, the quandle coloring invariant.

- We can extract some information from the derivation polynomial of a tame knot with respect to a finite quandle.
- Let K be a tame knot with the derivation polynomial $D_X(K)(u) = a_0 + a_1 u + \cdots + a_n u^n$ with respect to a finite quandle X.
- Then the constant term a_0 is the quandle coloring invariant, which corresponds to the trivial action of $Q(K)$ on X.
- For each $k \geq 1$, the coefficient a_k counts the number of non-trivial quandle actions ϕ of $Q(K)$ on X for which $|\operatorname{Der}_\phi\left(Q(K), X\right)| = k - 1$.

Proposition

The derivation polynomial of a tame knot is a proper enhancement of the quandle coloring invariant.

- Let X be a quandle with matrix

$$\begin{bmatrix} 1 & 3 & 5 & 7 & 9 & 11 & 2 & 4 & 6 & 8 & 10 \\ 11 & 2 & 4 & 6 & 8 & 10 & 1 & 3 & 5 & 7 & 9 \\ 10 & 1 & 3 & 5 & 7 & 9 & 11 & 2 & 4 & 6 & 8 \\ 9 & 11 & 2 & 4 & 6 & 8 & 10 & 1 & 3 & 5 & 7 \\ 8 & 10 & 1 & 3 & 5 & 7 & 9 & 11 & 2 & 4 & 6 \\ 7 & 9 & 11 & 2 & 4 & 6 & 8 & 10 & 1 & 3 & 5 \\ 6 & 8 & 10 & 1 & 3 & 5 & 7 & 9 & 11 & 2 & 4 \\ 5 & 7 & 9 & 11 & 2 & 4 & 6 & 8 & 10 & 1 & 3 \\ 4 & 6 & 8 & 10 & 1 & 3 & 5 & 7 & 9 & 11 & 2 \\ 3 & 5 & 7 & 9 & 11 & 2 & 4 & 6 & 8 & 10 & 1 \\ 2 & 4 & 6 & 8 & 10 & 1 & 3 & 5 & 7 & 9 & 11 \end{bmatrix}.$$

- Consider the knots 4_1 and 5_2. A GAP computation yields
$$|\operatorname{Hom}\big(Q(4_1), X\big)| = |\operatorname{Hom}\big(Q(5_2), X\big)| = 11$$
and
$$|\operatorname{Hom}\big(Q(4_1), \operatorname{Conj}(\operatorname{Aut}(X))\big)| = |\operatorname{Hom}\big(Q(5_2), \operatorname{Conj}(\operatorname{Aut}(X))\big)| = 330.$$
Thus, coloring by the quandles X and $\operatorname{Conj}\big(\operatorname{Aut}(X)\big)$ do not distinguishes the knots 4_1 and 5_2.
- Since $|X| = 11$, both the knots have only trivial colorings by X, and
$$\operatorname{Hom}\big(Q(4_1), X\big) \cong \operatorname{Hom}\big(Q(5_2), X\big) \cong X.$$
Thus, the hom quandle invariant of Crans-Nelson does not distinguish the knots 4_1 and 5_2.
- The derivation polynomials are
$$D_X(5_2)(u) = 11 + 120u + 209u^2$$
and
$$D_X(4_1)(u) = 11 + 230u + 99u^2,$$
respectively, and hence distinguishes the knots.
- In fact, the total derivation quandles $\mathcal{D}\big(Q(5_2), X\big)$ and $\mathcal{D}\big(Q(4_1), X\big)$ have sizes 220 and 110, respectively.

- It is interesting to find new properties of knot quandles.
- The notion of residual finiteness (and other residual properties) of groups plays a crucial role in combinatorial group theory and low dimensional topology.
- A group G is called residually finite if for each $g \in G$ with $g \neq 1$, there exists a finite group F and a homomorphism $\phi : G \to F$ such that $\phi(g) \neq 1$.
- Equivalently, G is residually finite if and only if for $g, h \in G$ with $g \neq h$, there exists a finite group F and a homomorphism $\phi : G \to F$ such that $\phi(g) \neq \phi(h)$.

- The preceding observation motivates the following definition.

Definition

A quandle X is said to be residually finite if for all $x, y \in X$ with $x \neq y$, there exists a finite quandle F and quandle homomorphism $\phi : X \to F$ such that $\phi(x) \neq \phi(y)$.

- Every trivial quandle is residually finite.
- Every free quandle is residually finite [BSS, 2018].

Theorem [BSS, 2018]

The knot quandle of a tame knot is residually finite.

We outline a proof now.

- Let H be a subgroup of a group G, G/H the set of right cosets of H in G and $z \in \mathrm{Z}(H)$ a fixed element.
- G/H with the binary operation

$$\bar{x} * \bar{y} = \bar{x}(\bar{y}^{-1}\bar{z}\bar{y})$$

  for $\bar{x}, \bar{y} \in G/H$ forms a quandle, denoted $(G/H, z)$.
- A subgroup H of a group G is said to be finitely separable in G if for each $g \in G \setminus H$, there exists a finite group F and a group homomorphism $\phi : G \to F$ such that $\phi(g) \notin \phi(H)$.

Proposition-II

Let H be a subgroup of a group G. If H is finitely separable in G, then the quandle $(G/H, z)$ is residually finite.

- Long and Niblo proved the following using the fact that doubling a 3-manifold along its boundary preserves residual finiteness.

Theorem [Long-Niblo, 1991]

Let M be an orientable, irreducible compact 3-manifold and X an incompressible connected subsurface of a component of $\partial(M)$. If $p \in X$ is a base point, then $\pi_1(X, p)$ is a finitely separable subgroup of $\pi_1(M, p)$.

- Let $V(K)$ be a tubular neighbourhood of a knot K in $\mathbb{S}^3$. Then the knot complement $C(K) := \overline{\mathbb{S}^3 \setminus V(K)}$ has boundary $\partial C(K)$ a torus.
- Let $x_0 \in \partial C(K)$,

$$\iota_* : \pi_1\big(\partial C(K), x_0\big) \longrightarrow \pi_1\big(C(K), x_0\big)$$

the homomorphism induced by the inclusion, and $P := \iota_*\big(\pi_1(\partial C(K), x_0)\big)$ the peripheral subgroup of the knot group.

Corollary-III

The peripheral subgroup of a non-trivial tame knot is finitely separable in the knot group.

- By constructing a transitive action of the knot group of a tame knot on its knot quandle , Joyce proved the following:

Proposition-IV

Let K be a tame knot with knot group G and knot quandle $Q(K)$. Let P be the peripheral subgroup of G containing the meridian m. Then $Q(K) \cong (G/P, m)$.

- We can now prove the main result.

  Proof: Let K be a tame knot. If K is an unknot, then the knot quandle $Q(K)$ is vacuously residually finite being a trivial quandle with one element. If K is non-trivial, then using Proposition-II, Corollary-III and Proposition-IV, it follows that $Q(K)$ is residually finite.

- Joyce's proof of the complete invariance of the knot quandle (up to orientation) of a tame knot depends heavily on the following result.

Theorem [Waldhausen [1968]]

Let M and N be 3-manifold which are irreducible and boundary irreducible. Let $\psi : \pi_1(M) \to \pi_1(N)$ be an isomorphism which respects the peripheral structure. Then there exists a homeomorphism $f : M \to N$ which induces ψ.

- There are tame links whose complements in $\mathbb{S}^3$ are reducible 3-manifolds. Thus, quandles associated to tame links are not complete invariants.
- For the same reason, Theorem of Long and Niblo is not applicable for tame links.

- If L_n is a trivial n-component link, then the link quandle $Q(L_n)$ is isomorphic to the free quandle on n generators which is residually finite [BSS, 2018].

Question

Let L be a tame link with more than one component. Is the link quandle $Q(L)$ residually finite?

Спасибо за ваше внимание

Thank you for your attention