CYCLICALLY PRESENTED SIERADSKI GROUPS WITH EVEN NUMBER OF GENERATORS AND 3-MANIFOLDS

T. Kozlovskaya

Tomsk State University
Regional Scientific and Educational Mathematical Center

25 June 2019

Three dimensional manifolds

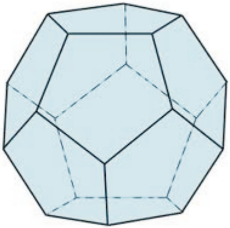
– closed orientable 3-manifolds

– study the problem if a given presentation of group is geometric

– construct manifolds which are n-fold cyclic branched coverings of lens spaces

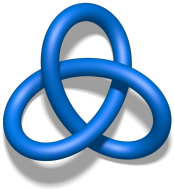
Spherical and hyperbolic dodecahedral spaces

The spherical $(\frac{2\pi}{3})$ and hyperbolic $(\frac{2\pi}{5})$ dodecahedra.



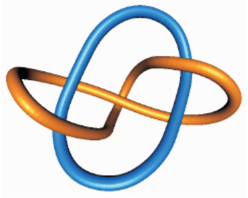
Seifert H., Weber C (1933) the 3 – manifolds (spherical and hyperbolic) by a pairwise identification of faces of the regular dodecahedra.

Topological property of dodecahedral hyperbolic space



The Seifert–Weber dodecahedral spherical manifold is the 3 – fold cyclic branched covering of the 3 – sphere branched over the trefoil knot

The Seifert–Weber dodecahedral hyperbolic manifold is the 5 – fold cyclic branched covering of the 3 – sphere branched over the Whitehead link



Cyclically presented groups

Let $\mathbb{F}$ be the free group of rank $m \geq 1$ with generators $x_1, x_2, \ldots, x_m$ and let $w = w(x_1, x_2, \ldots, x_m)$ be a cyclically reduced word in $\mathbb{F}_m$. Let $\eta : \mathbb{F}_m \to F_m$ be an automorphism given by $\eta(x_i) = x_{i+1}$, $i = 1, \ldots, m-1$, and $\eta(x_m) = x_1$. The presentation

$$G_m(w) \; = \; \langle x_1, \ldots, x_m \mid w = 1, \, \eta(w) = 1, \, \ldots, \, \eta^{m-1}(w) = 1 \rangle,$$

is called an m-cyclic presentation with defining word w. A group G is said to be cyclically presented group if G is isomorphic $G_m(w)$ for some m and w.

Sieradski groups

Cyclically presented groups

$$S(m) = \langle x_1, x_2, \ldots, x_m \mid x_i x_{i+2} = x_{i+1}, \quad i = 1, \ldots m \rangle,$$

where all subscripts are taken by mod m, were called the Sieradski groups. The groups

$$S(m,p,q) = \langle x_1, \ldots, x_m \mid x_i\, x_{i+q} \cdots x_{i+(q-1)dq-q}\, x_{i+(q-1)dq} = x_{i+1}\, x_{i+q+1} \cdots x_{i+(q-1)dq-q+1}, \quad i = 1, \ldots, m \rangle,$$

are called the generalised Sieradski groups. All subscripts are taken by mod m. Parameters p and q are co-prime integers such that $p = 1 + dq$, $d \in \mathbb{Z}$.

Examples of manifolds with cyclic symmetry

The Sieradski manifolds are the n-fold cyclic coverings of S^3 branched over the trefoil knot.

Cavicchioli A., Hegenbarth F. and Kim A.C. (1999):The cyclic presentation $S(m, p, q)$ corresponds to a spine of the m-fold cyclic covering of the 3-sphere S^3 branched over the torus knot $T(p, q)$.

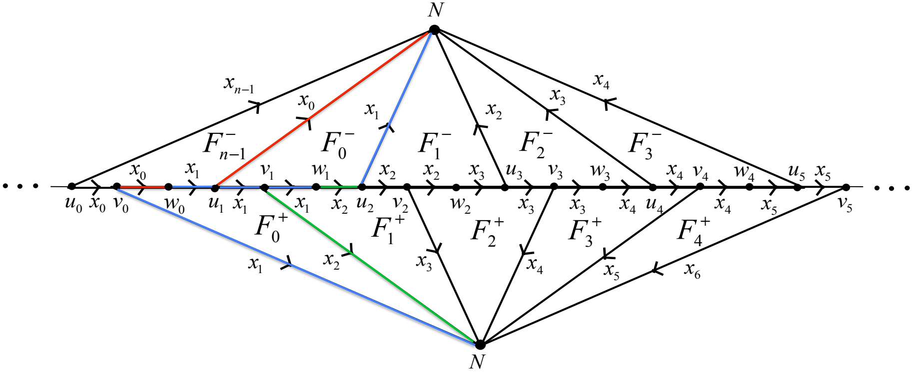
Cyclic presentations of $S(2n,3,2)$ with n generators

$$
\begin{aligned}
S(2n,3,2) &= G_{2n}(x_1x_3x_2^{-1}) \\
&= \langle x_1,x_2,\ldots,x_{2n} \,|\, x_ix_{i+2}=x_{i+1}, \quad i=1,\ldots,2n\rangle \\
&= \langle x_1,x_2,\ldots,x_{2n} \,|\, x_{2j}x_{2j+2}=x_{2j+1}, \quad x_{2j+1}x_{2j+3}=x_{2j+2} \quad \rangle \\
&= \langle x_2,x_4,\ldots,x_{2n} \,|\, (x_{2j}x_{2j+2})(x_{2j+2}x_{2j+4})=x_{2j+2} \quad \rangle \\
&= \langle y_1,y_2,\ldots,y_n \,|\, y_jy_{j+1}^2y_{j+2}=y_{j+1}, \quad j=1,\ldots,n\rangle \\
&= G_n(y_1y_2^2y_3y_2^{-1}).
\end{aligned}
$$

$S(2n,3,2)$ is the fundamental group of the 3-manifold $\mathcal{B}(2n,3,2)$, which is the $2n$-fold cyclic covering of S^3 branched over the trefoil knot.

It is natural to ask: if the cyclic presentation $G_n(x_1x_2^2x_3x_2^{-1})$ is geometric too?

J. Howie, G. Williams, Fibonacci type presentations and 3-manifolds, Topology Appl. 215 (2017), 24–34.

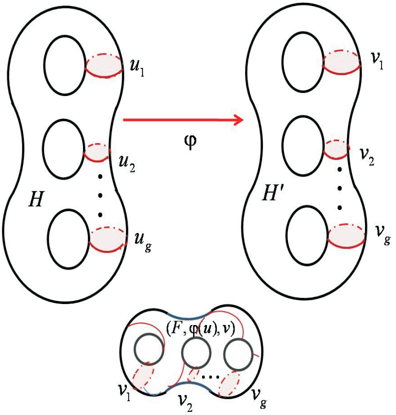


Theorem (J. Howie, G. Williams)

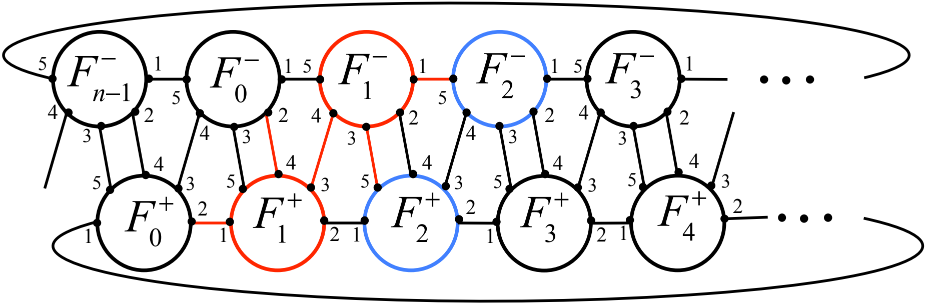
Cyclic presentation $G_n(x_0x_1^2x_2x_1^{-1})$ *is geometric.*

Heegaard diagram

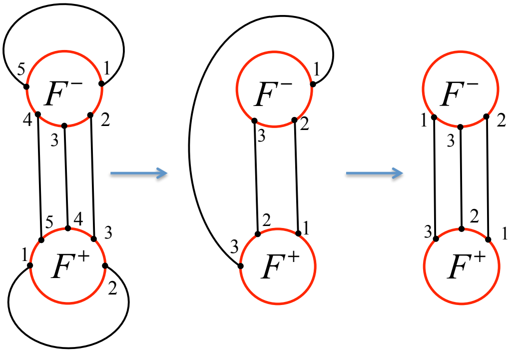
Let $M = H \cup H^{'}$ is a genus g Heegaard splitting of a manifold M, $u = u_1, \ldots, u_g$ and $v = v_1, \ldots, v_g$ are meridian systems for H and $H^{'}$ and $F = \partial H = \partial H^{'}$ is a Heegaard surface. Let $\varphi : F \to F$ be homeomorphism of their boundaries. Then the triple $(F, \varphi(u), v)$ is called a Heegaard diagram of M.



Heegaard diagram of the manifold $\mathcal{S}(2n, 3, 2)$



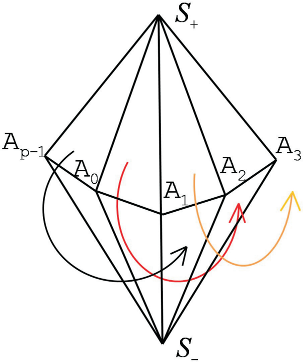
Simplifying a Heegaard diagram of $\mathcal{S}(2n, 3, 2)/\rho$.

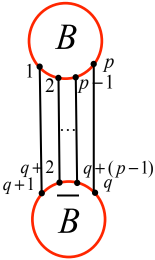


Lens space

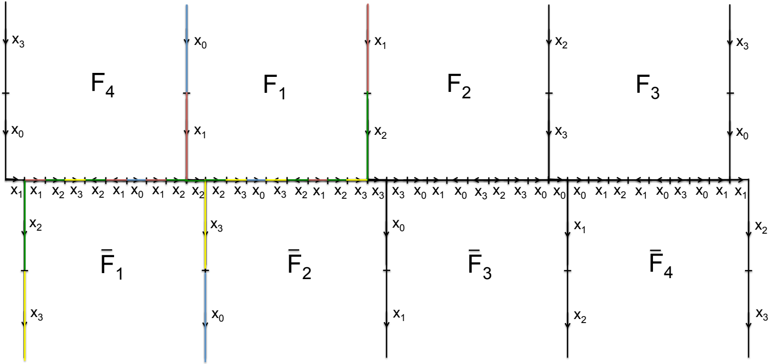
Let $p \geqslant 3$, $0 < q < p$ and $(p, q) = 1$.

Consider a p-gonal bipyramid, i.e. the union of two cones over a regular p-gon, where the vertices of the p-gon are denoted by $A_0, A_1, \ldots, A_{p-1}$ and apex of cones are denoted by S_+ and S_-. For each i we glue the face $A_i S_+ A_{i+1}$ with the face $A_{i+q} S_- A_{i+q+1}$. The manifold obtained is the lens space $L_{p,q}$





If the cyclic presentation $G_n(x_0x_1x_2x_2x_3x_4x_3^{-1}x_2^{-1}x_1x_2x_3x_2^{-1}x_1^{-1})$ is *geometric* ?

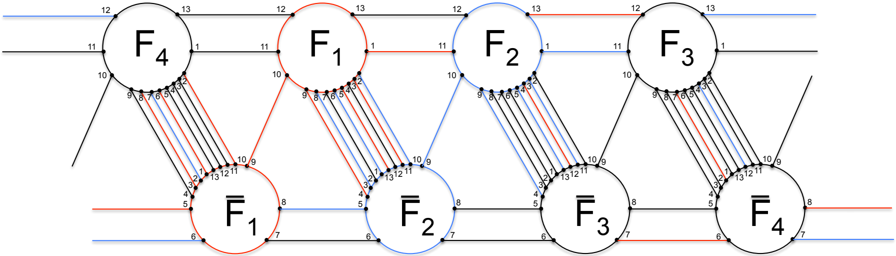


$$x_0x_1x_2x_2x_3x_0x_3^{-1}x_2^{-1}x_1x_2x_3x_2^{-1}x_1^{-1} = 1,$$

Heegaard diagram for the case $n = 4, (p, q) = (5, 2)$

Theorem (K.- Vesnin A.)

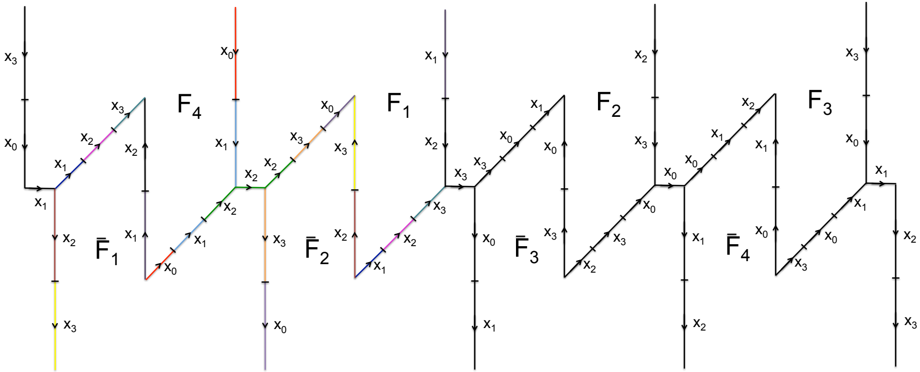
The cyclic presentation $G_n(x_0x_1x_2x_2x_3x_4x_3^{-1}x_2^{-1}x_1x_2x_3x_2^{-1}x_1^{-1})$ is geometric, i. e. it corresponds to a spine of a closed 3-manifold.



For each n the manifold from theorem is an n-fold cyclic covering of $L(5, 1)$.

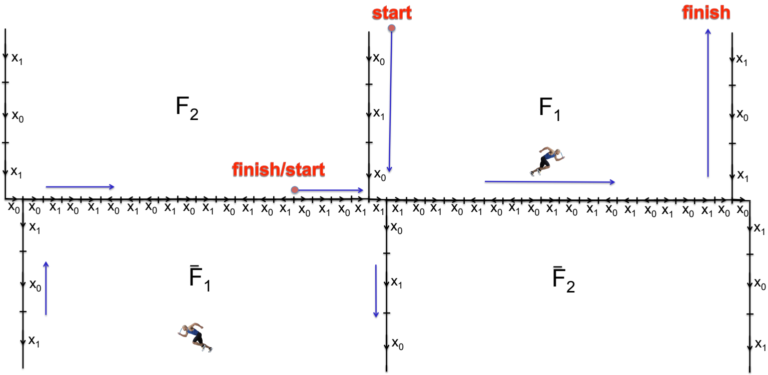
Three-manifold Recognizer:
$\mathcal{S}(8, 5, 2)$ is the Seifert manifold $(S^2, (4, 1), (5, 2), (5, 2), (1, -1))$.

The complex for the case $n = 4, (p, q) = (5, 2)$



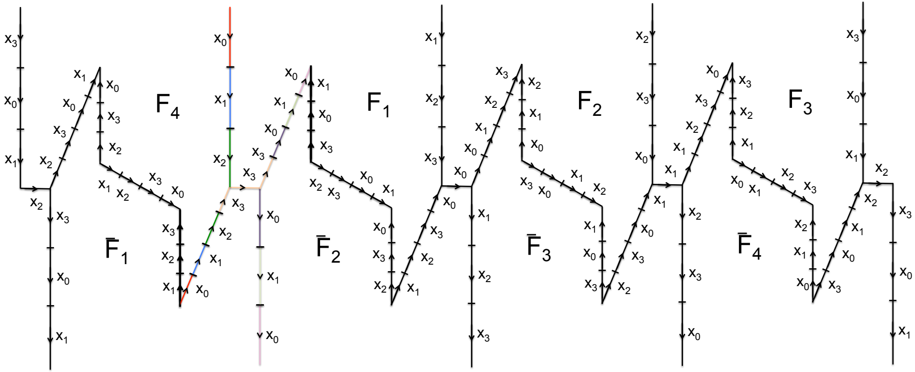
$$x_0 x_1 x_2 x_2 x_3 x_0 x_3^{-1} x_2^{-1} x_1 x_2 x_3 x_2^{-1} x_1^{-1} = 1,$$

The complex for the case $n = 1, (p, q) = (7, 2)$



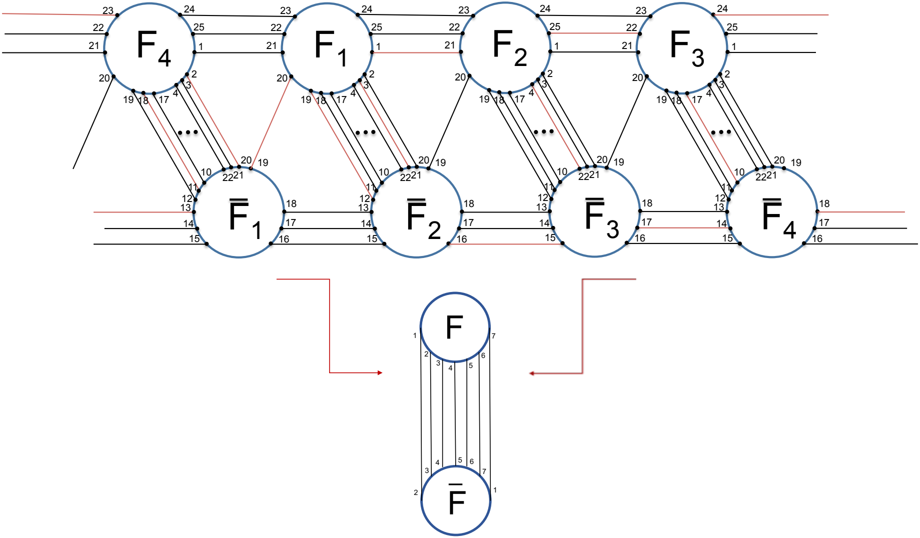
$$x_0x_1x_0x_1x_1x_0x_1x_0x_1^{-1}x_0^{-1}x_1^{-1}x_0x_1x_0x_1x_0^{-1}x_1^{-1}x_0^{-1}x_1x_0x_1x_0x_1^{-1}x_0^{-1}x_1^{-1} = 1$$
$$x_1x_0x_1x_0x_0x_1x_0x_1x_0^{-1}x_1^{-1}x_0^{-1}x_1x_0x_1x_0x_1^{-1}x_0^{-1}x_1^{-1}x_0x_1x_0x_1x_0^{-1}x_1^{-1}x_0^{-1} = 1$$

The complex for the case $n = 4, (p, q) = (7, 2)$



$$x_0x_1x_2x_3x_3x_0x_1x_2x_1^{-1}x_0^{-1}x_3^{-1}x_2x_3x_0x_1x_0^{-1}x_3^{-1}x_2^{-1}x_1x_2x_3x_0x_3^{-1}x_2^{-1}x_1^{-1} = 1$$
$$x_1x_2x_3x_0x_0x_1x_2x_3x_2^{-1}x_1^{-1}x_0^{-1}x_3x_0x_1x_2x_1^{-1}x_0^{-1}x_3^{-1}x_2x_3x_0x_1x_0^{-1}x_3^{-1}x_2^{-1} = 1$$
$$x_2x_3x_0x_1x_1x_2x_3x_0x_3^{-1}x_2^{-1}x_1^{-1}x_0x_1x_2x_3x_2^{-1}x_1^{-1}x_0^{-1}x_3x_0x_1x_2x_1^{-1}x_0^{-1}x_3^{-1} = 1$$
$$x_3x_0x_1x_2x_2x_3x_0x_1x_0^{-1}x_3^{-1}x_2^{-1}x_1x_2x_3x_0x_3^{-1}x_2^{-1}x_1^{-1}x_0x_1x_2x_3x_2^{-1}x_1^{-1}x_0^{-1} = 1$$

Heegaard diagram for the case $n = 4, (p, q) = (7, 2)$



The generalised Sieradski group $S(2n, 7, 2)$

Theorem

For $n \geqslant 1$ *group* $S(2n, 7, 2)$ *has a presentation with n generators* $y_0, y_1, \ldots, y_{n-1}$ *and defining relations*

$$\{y_i y_{i+1} y_{i+2} y_{i+3} y_{i+3} y_{i+4} y_{i+5} y_i {y_{i+5}}^{-1} y_{i+4}^{-1} y_{I+3}^{-1} y_{i+2} y_{i+3} y_{i+4}$$
$$y_{i+5} y_{i+4}^{-1} y_{i+3}^{-1} y_{i+2}^{-1} y_{i+1} y_{i+2} y_{i+3} y_{i+4} y_{i+3}^{-1} y_{i+2}^{-1} y_{y+1}^{-1} = 1 \quad i = 0, \ldots, n-1.\}$$

These presentation is geometric and corrisponds to n-fold cyclic branched covering of the lens space $L(7, 1)$.

Three-manifold Recognizer:

$S(2, 7, 2)$ is the Seifert manifold $(S^2, (2, 1), (7, 2), (7, 2), (1, -1))$
$S(6, 7, 2)$ is the Seifert manifold $(S^2, (6, 1), (7, 3), (7, 3), (1, -1))$.

THANK YOU