FULLY INERTLY SOCLE-REGULAR ABELIAN p-GROUPS

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Abstract. We define the so-called *inertly fully transitive* and *fully inertly socle-regular* Abelian p-groups and study them in a comprehensive way with respect to their crucial properties. The achieved results somewhat continue recent investigations due to the second named author and Goldsmith in Arch. Math. Basel (2009) and J. Algebra (2010), respectively.

1. Introduction and Backgrounds

Throughout the paper, all groups into consideration, unless specified something else, are assumed to be Abelian and p-torsion, where p is a prime fixed for the duration. Almost all used terminology and notations are classical as the unexplained explicitly ones follow those from [11] and [12]. For instance, for any prime p, the symbol $G[p^n] = \{g \in G \ : \ p^n g = 0\}$ denotes the p^n*-socle* of the group G, and the symbol $p^nG = \{p^n g \ : \ g \in G\}$ denotes the *n-th power subgroup* of G, where $n \in \mathbb{N}$. Inductively, for any ordinal α, $p^\alpha G = p(p^{\alpha-1}G)$ when $\alpha - 1$ exists or $p^\alpha G = \cap_{\beta<\alpha} p^\beta G$ for otherwise. For an element

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x of a group G, the letter $U_G(x)$ stands for the standard Ulm sequences of heights of x as computed in G.

Recall that the two subgroups H, K of a group G are said to be *commensurable* if the intersection $H \cap K$ has finite index simultaneously in H and in K. So, the subgroup H of a group G is called *fully inert* if the factor-group $(H+\varphi(H))/H$ is finite, i.e., H is commensurable with $H + \varphi(H)$ for any $\varphi \in \mathrm{E}(G)$. Fully inert subgroups are studied intensively in [1], [3, 4], [8], [9], [10] and [13], respectively.

Recall also that the least ordinal number τ equipped with the property $p^{\tau+1}G = p^{\tau}G$ it is said to be the *length* of the group G. Clearly, $p^{\tau}G = 0$ provided that G is reduced.

We now come to our first critical point of view.

Definition 1.1. A group G is said to be *inertly fully transitive* if, for any its non-zero subgroup H with finite quotient G/H, and for any two elements x in H and y in G with $U_G(x) \leq U_G(y)$, there is a homomorphism $f : H \to G$ such that $f(x) = y$.

In that aspect, let us recollect now the well-known concept that a group G is said to be *fully transitive* if, for any two elements x, y of G with $U_G(x) \leq U_G(y)$, there is an endomorphism $f : G \to G$ such that $f(x) = y$.

The next new notion is our basic tool.

Definition 1.2. A group G of length τ is said to be *fully inertly socle-regular* if, for every infinite fully inert subgroup H of G, there exists an ordinal σ depending on H such that $\big(H[p] \cap p^\sigma G\big) + F = (p^\sigma G)[p]$ for some finite subgroup F of $(p^\sigma G)[p]$, where $\sigma < \tau$ if $p^\tau G = 0$ and $\sigma \leq \tau$ otherwise. Then $\big(H[p] \cap p^\sigma G\big) \oplus F_\sigma = (p^\sigma G)[p]$ for some subgroup $F_\sigma \leq F$.

In that direction, let us recollect now that a group G is said to be *socle-regular* in [6] and [7] if, for each fully invariant subgroup M of G, there exists an ordinal σ depending on M such that $M[p] = (p^\sigma G)[p]$. It is clear that if $H[p]$ is commensurable with the fully invariant subgroup K of the socle-regular group G, then $H[p]$ is commensurable with some $(p^\sigma G)[p]$. In fact, noticing that $f(H[p])$ is commensurable with $f(K)$ for all $f \in \mathrm{E}(G)$, one deduces that $p(H[p]) = 0$

is commensurable with pK, and then pK is finite and H is commensurable with $K[p]$.

Since in the direct sums of cyclic p-groups all fully inert subgroups are commensurable with fully invariant subgroups (see, e.g., [13]), then according to the fact that these direct sums are also socle-regular (cf. [6]), we can get following useful observation that *the direct sum of cyclic p-groups are fully inertly socle-regular.*

We, moreover, can indicate that if $\big(H[p] \cap p^{\sigma}G\big) \oplus F_{\sigma} = (p^{\sigma}G)[p]$, then $\big(H[p] \cap p^{\alpha}G\big) \oplus F_{\alpha} = (p^{\alpha}G)[p]$ for any $\alpha \geq \sigma$ and some $F_{\alpha} \leq (p^{\alpha}G)[p]$. Indeed, this follows from the fact that $H[p] \cap p^{\sigma}G$ has finite index in $(p^{\sigma}G)[p]$. Consequently, in this case there exists a minimal ordinal number σ_0 such that $\big(H[p] \cap p^{\sigma_0}G\big) \oplus F_{\sigma_0} = (p^{\sigma_0}G)[p]$. **Hereafter, for the remainder of the article, we will fix this indicator** σ_0.

Besides, for a reduced group G, it must be that $\big(H[p] \cap p^{\sigma}G\big) \oplus F_{\sigma} = (p^{\sigma}G)[p]$ for some fully inert subgroup H and $F_{\sigma} \neq 0$ for all ordinal numbers $\sigma < \tau$. In fact, let $\tau = \alpha + 1$ and $G[p] = F \oplus (p^{\alpha}G)[p]$ for some $F \leq G[p]$. We can take

H of the form $H = F \oplus F'$, where F' is a proper subgroup in $(p^\alpha G)[p]$ of finite index. This substantiates our claim after all.

We can slightly strengthen the aforementioned notion of *full inert socle-regularity* to the following one:

Definition 1.3. A group G is *cleanly fully inertly socle-regular* if, for each its fully inert subgroup H, there exists an ordinal σ which depends on H with the property that $H[p] = (p^\sigma G)[p]$.

One observes at first glance that $H[p] = (p^\sigma G)[p]$ implies that $H[p] \cap p^\sigma G = (p^\sigma G)[p]$ and thus $(H[p] \cap p^\sigma G) + F = (p^\sigma G)[p]$ for any finite subgroup F of $(p^\sigma G)[p]$, while it is hardly expected that the second equality will imply the first one – just it yields that $(p^\sigma G)[p] \subseteq H[p]$.

A detailed exploration of the introduced above concepts will be given in the next subsequent section.

2. Preliminary and Main Results

We start here with some helpful technicalities:

Lemma 2.1. *If H is a fully inert subgroup of a group G, then $K = H[n]$ is also a fully inert subgroup of G for every $n \in \mathbb{N}$.*

Proof. One sees that $\big(K+\varphi(K)+H\big)/H \cong \big(K+\varphi(K)\big)/\big(H\cap (K+\varphi(K))\big)$ and $H \cap (K + \varphi(K)) = K$. So, the quotient $(K + \varphi(K))/K$ is finite, as required. □

This statement can also be easily deduced and from the fact that the intersection of two fully inert subgroups is also fully inert.

Lemma 2.2. [3, Lemma 3] *Suppose H is a fully inert subgroup of the group $G = A \oplus B$, and $\pi\colon G \to A$, $\theta\colon G \to B$ are the corresponding projections. Then the subgroup $H \cap A$ is fully inert in A, the subgroups $(H \cap A) \oplus (H \cap B)$ and $\pi H \oplus \theta H$ are commensurable with H, and if $\varphi \in Hom(B, A)$, then the subgroup $H \cap A + \varphi(H \cap B)$ is commensurable with $H \cap A$.*

After that, we are in a position to proceed by proving with

Lemma 2.3. *Let H be a fully inert subgroup of the group $G = \bigoplus_{i\in I} G_i$, where the index set I is infinite. Then H is* *commensurable with $\bigoplus_{i\in I}(H\cap G_i)$, where $H\cap G_i$ are fully inert in G_i and even almost all $H\cap G_i$ are fully invariant in G_i.*

Proof. Letting $\pi_i\colon G\to G_i$ be the projections, it follows from [4, Lemma 7] that H is commensurable with $\sum_{i\in I}\pi_i(H)$. Consequently, $H\cap G_i=\pi_i(H)$ for almost all $i\in I$. But each $H\cap G_i$ is commensurable with $\pi_i(H)$, so that H is commensurable with $\bigoplus_{i\in I}(H\cap G_i)$ and each $H\cap G_i$ is fully inert in G_i. If we now assume that $H\cap G_j$ is not fully invariant in G_j for each index j from an infinite subset $J\subseteq I$, then exists $\varphi_j\in \mathrm{E}(G_j)$ with $\varphi_j(H\cap G_j)\not\leq H\cap G_j$. Setting $\varphi=1+\bigoplus_{j\in J}\varphi_j$, we then obtain that $|\,(H+\varphi(H))/H\,|\geq\aleph_0$, as required. □

Let us now B be a basic subgroup of the group G writing $B=B_1\oplus B_2\oplus\dots$, where $B_n\cong\big(\mathbb{Z}(p^n)\big)^{(\alpha_n)}$ for some cardinal numbers α_n, and let $B_n^*=B_{n+1}\oplus B_{n+2}\oplus\dots$. Then $G=B_1\oplus\cdots\oplus B_n\oplus G_n^*$, where $G_n^*=B_n^*+p^nG$ (see, e.g., [11, Theorem 32.4]).

With Lemma 2.2 at hand, it follows directly the following.

Lemma 2.4. *Let $G = D \oplus A$, where D is a divisible group and A is a reduced group. Then the subgroup $H \leq G[p]$ is fully inert if, and only if, H is commensurable with $(H \cap D) \oplus (H \cap A)$, where $H \cap D$ has finite index in $D[p]$ and $H \cap A$ is fully inert in A.*

Since $D[p] \leq p^n G$ for all $n \in \mathbb{N}$, it follows from Lemma 2.4 that the study of fully inertly socle-regular groups is reducible to reduced groups. That is why, we shall assume that all further groups are themselves *reduced*.

Lemma 2.5. *Let H be a fully inert subgroup of a group G.*

(1) If $n \in \mathbb{N}$ is chosen such that $|\, H[p] \cap B_n \,| \geq \aleph_0$, then $(H[p] \cap p^{n-1}G) + F = (p^{n-1}G)[p]$ for some finite subgroup F, i.e., $\sigma_0 \leq n-1$.

(2) If G is an unbounded group and $|\, H[p]/(H[p] \cap p^{\omega}G) \,| \geq \aleph_0$, then it is possible to choose subgroups B_n such that $H \cap B_n \neq 0$ for endless ??? numbers $n \in \mathbb{N}$.

(3) If H is an infinite fully inert subgroup of G and $\sigma_0 = n \in \mathbb{N}$, then $H[p]$ is commensurable with $(p^n G)[p]$.

Proof. (1) We shall show that $(H[p] \cap B_m) + K_m = B_m[p]$ for all $m \geq n$ and some finite subgroup K_m of $B_m[p]$ as, moreover, $K_m = 0$ for almost all $m \geq n$, provided G is unbounded. to that aim, for every $k \in \mathbb{N}$ we have $B_k[p] = (H[p] \cap B_k) \oplus B'_k$ for some $B'_k \leq B_k[p]$. Select in B_n the direct summand C such that $C[p] = H[p] \cap B_n$. If $|\, B'_m[p] \,| \geq \aleph_0$ for some $m \geq n$, then there exists a homomorphism $f \colon C \to B_m$ such that $f(C[p]) \leq B'_m[p]$ and $|\, f(C[p]) \,| \geq \aleph_0$. Thus $|\, (H + f(H))/H \,| \geq \aleph_0$, a contradiction. But if $B'_{k_s} \neq 0$ for $n \leq k_1 < k_2 < \ldots$, then there exists a homomorphism $f \colon C \to B_{k_1} \oplus B_{k_2} \oplus \ldots$ with the property that $|\, f(C[p]) \,| \geq \aleph_0$, whence we once again get a contradiction. Note that $(p^{n-1}G)[p] = G^*_{n-1}[p] = B_n[p] \oplus G^*_n[p]$ and that either the p-height of every element from $G^*_n[p]$ is greater than the p-height of all non-zero elements from $B_n[p]$ or as we earlier deduce that both $H[p] \cap B_n$ and $H[p] \cap G^*_n[p]$ have finite indexes in B_n and $G^*_n[p]$, respectively, thus getting the truthfulness of our statement.

(2) We have $H[p] = H' \oplus (H[p] \cap p^\omega G)$ and $|\, H' \,| \geq \aleph_0$. If $|\, H' \cap B_n \,| \geq \aleph_0$ for some n, then by point (1) the statement follows immediately. If not, then according to [11, Corollary

27.2], any element of order p and finite height can be embedded in a cyclic direct summand, which is a direct summand of some B_n. Since H' is infinite and $|\,H' \cap B_n\,| < \aleph_0$ for all n, then such direct summands B_n must be, definitely, infinitely many.

(3) It follows with the aid of item (1). □

As it will be hopefully seen below in Example 2.11, this cannot be happened in the case of infinite ordinal σ_0.

Theorem 2.6. *Let G be a torsion-complete group and H its infinite fully inert subgroup. Then G is cleanly fully inertly socle-regular and, specifically, $H[p] = (p^{n-1}G)[p]$ for some $n \in \mathbb{N}$.*

Proof. Since H is infinite and G is separable, $H[p]$ is also infinite. If foremost $|\,H[p] \cap B_n\,| \geq \aleph_0$ for some natural number n, then by Lemma 2.5 the statement is true. Assume now that $|\,H[p] \cap B_k\,| < \aleph_0$ for all $k \in \mathbb{N}$. Therefore, Lemma 2.2 leads to the fact that B is unbounded. Since $H[p]$ is infinite, in accordance with Lemma 2.5 there exists a sequence $k_1 < k_2 < \ldots$ and non-zero direct summands B'_{k_i} of B_{k_i} such that $B' = B'_{k_1} \oplus B'_{k_2} \oplus \ldots$, where $B'[p] \leq H$.

Suppose $(p^nG)[p] = (H[p] \cap p^nG) \oplus L_n$. Assume that for all naturals n the subgroups L_n are infinite. So, for any $b_i \in B'_{k_i}[p]$ there exists an element $g_i \in L_{n_i}$ with $h(g_i) \geq h(b_i)$. We now have that $g_i = x_i + y_i$, where $x_i \in H[p] \cap pG$ and $y_i \in L_1$. Assume also that $\langle g_1, \ldots, g_m \rangle \cap H = 0$. Since $L_{n_{m+1}}$ is infinite, then there exists an element $g_{m+1} \in L_{n_{m+1}}$ such that $\langle g_1, \ldots, g_m, g_{m+1} \rangle \cap H = 0$. In fact, the condition $\langle g_1, \ldots, g_m, g_{m+1} \rangle \cap H \neq 0$ for any $g_{m+1} \in L_{n_{m+1}}$ means that $y_{m+1} \in \langle y_1, \ldots, y_m \rangle$, which contradicts to the infinity of $L_{n_{m+1}}$. Thus there exists a homomorphism $f \colon B \to G$ with $f(b_i) = g_i$. Since either $G = B$ or G is an unbounded torsion-complete group, it is possible to get that $f \in \mathrm{E}(G)$. Finally, $|\,(H[p] + f(H[p]))/H[p]\,| \geq \aleph_0$, as needed. □

let us recollect once again that each separable group is necessarily socle-regular groups. The next construction shows that there are socle-regular groups which need not be fully inertly socle-regular.

Example 2.7. There exist non fully inertly socle-regular separable groups.

Proof. According to [5], there exists a separable group G for which $\mathrm{E}(G) = \mathbb{Q}_p^* \oplus \mathrm{E}_s(G)$, where $\mathbb{Q}_p^*$ is the ring of p-adic integers, and $\mathrm{E}_s(G)$ is the ideal of small endomorphisms of the endomorphism ring $\mathrm{E}(G)$. Since $\mathrm{E}(G) \neq \mathrm{E}_s(G)$, the group G is manifestly unbounded. Take such an increasing sequence of natural numbers $k_1 < k_2 < \dots$ that between any two ones $k_i < k_{i+1}$ there exists n_i with $B_{n_i} \neq 0$. Moreover, in any $B_{k_i}[p]$ take a cyclic subgroup, and let H be their direct sum. If $\psi \in \mathrm{E}(G)$, then $\psi = f + \varphi$, where f acts as multiplication on p-adic integer number while φ is a small endomorphism. Therefore, $(p^n G)[p] \leq \ker \varphi$ for some $n \in \mathbb{N}$, so that $\varphi(H)$ is finite. Since f acts invariantly on all subgroups, H is definitely fully inert but it is not commensurable with all $(p^m G)[p]$, as expected. □

Proposition 2.8. *The group G is fully inertly socle-regular if, and only if, the direct sum $A = G^{(n)}$ is fully inertly socle-regular for any $n \in \mathbb{N}$.*

Proof. First, assume that G is fully inertly socle-regular. Writing A in the form $A = G_1 \oplus \cdots \oplus G_n$, where all $G_i \cong G$, we let

$H \leq A[p]$ be an infinite fully inert subgroup. Then, in view of Lemma 2.2, H is commensurable with $(G_1 \cap H) \oplus \cdots \oplus (G_n \cap H)$, where each $G_i \cap H$ is an infinite fully inert subgroup, so $(p^{\alpha_i} G_i) \cap H$ is commensurable with $(p^{\alpha_i} G_i)[p]$ for some α_i. If $\alpha_i < \alpha_j$, then in virtue of the isomorphisms $G_i \cong G$ and of the finiteness of the number n, the subgroup $(p^{\alpha_i} G)[p]$ is commensurable with $(p^{\alpha_j} G)[p]$, whence if α is one of $\alpha_1, \ldots, \alpha_n$, then $H[p]$ has to be commensurable with $(p^{\alpha} A)[p]$.

Conversely, suppose that A is fully inertly socle-regular and that $H \leq G[p]$ is a fully inert subgroup of G. Then, it is obvious that the subgroup $H^{(n)}$ is fully inert in A. Since the latter is fully inertly socle-regular, it must be that $H^n \cap (p^{\alpha} A)[p]$ is commensurable with $(p^{\alpha} A)[p]$ for some ordinal α. It follows immediately now that H is commensurable with $(p^{\alpha} G)[p]$ and thus G is fully inertly socle-regular, as asserted. □

The following statement is somewhat rather surprising, especially in its part (2) comparing it with the above Example 2.7.

Proposition 2.9. *(1) If the separable group G is fully inertly socle-regular, then the direct sum $A = G^{(k)}$ is fully* *inertly socle-regular for any finite or infinite cardinal number k.*

(2) If G is a separable group, then the direct sum $G^{(k)}$ is fully inertly socle-regular for any infinite cardinal number k.

(3) If the group $A = G^{(k)}$ is fully inertly socle-regular, where k is an infinite cardinal number, and in G each fully inert subgroup is commensurable with fully invariant subgroup, then G is fully inertly socle-regular.

Proof. According to Proposition 2.8, it suffices to consider the cardinal k to be infinite.

(1) Write A in the form $A = \bigoplus_{i\in I} G_i$, where all $G_i \cong G$. Letting $H \leq A[p]$ be an infinite fully inert subgroup, then by Lemma 2.2 the subgroup H is commensurable with $\bigoplus_{i\in I}(H \cap G_i)$. Since all G_i are isomorphic groups, there exists a finite subset $I_0 \subseteq I$ such that $H \cap G_j$ are also isomorphic, such that $H \cap G_j$ are fully invariant in G_j by Lemma 2.3 and such that $H \cap G_j$ is commensurable with $(p^n G_j)[p]$ for all $j \in J = I \setminus I_0$. It follows that $H \cap p^m G_j = (p^m G_j)[p]$ for some $m \geq n$. Indeed, if $x \in H \cap p^n G_j$ and $y \in (p^m G_j)[p] \setminus (H \cap p^m G_j)$, then there

exists a map $f \in \mathrm{E}(G_j)$ such that $f(x) = y$, that is in a sharp contrast with the full invariance of $H \cap p^n G_j$. Thus H is commensurable with $(p^m A)[p]$, as asked for.

(2) As already noticed above, separable groups are socle-regular by [6, Corollary 0.2], so this point follows directly from the proof of (1) as well as almost all of the subgroups $H \cap G_i$ are fully invariant.

(3) Let $H \leq G[p]$ be an infinite fully inert subgroup and Y is a fully invariant subgroup of G which is commensurable with H. Then, one checks that $Y^{(k)}$ is a fully invariant subgroup of A. It follows that $Y^{(k)}[p] \cap p^\sigma A$ is commensurable with $(p^\sigma A)[p]$. Moreover, as in (1), it can be proved that $Y^{(k)}[p] \cap p^\sigma A = (p^\sigma A)[p]$. We, consequently, deduce that $Y[p] \cap p^\sigma G = (p^\sigma G)[p]$ and that $H \cap p^\sigma G$ is commensurable with $(p^\sigma G)[p]$, as wanted. □

The next consequence is immediate.

Corollary 2.10. *If G is a torsion-complete group, then $G^{(k)}$ is a fully inertly socle-regular for every cardinal number k.*

The next two constructions unambiguously illustrate that for ordinals beyond ω the situations are rather complicated.

Example 2.11. (1) For every $n \in \mathbb{N}$ there exist groups of length $\omega + n$ which are not fully inertly socle-regular.

(2) For every $n \geq 2$ there exist groups G of length $\omega+n$ which have such fully invariant subgroups $H \leq G[p]$ that $H \leq G$ is not commensurable with $(p^\sigma G)[p]$ for any $\sigma < \tau = \omega + n$.

Proof. (1) Let $K = K_1 \oplus \cdots \oplus K_n$, where $K_i = Z(p^i)^{(\alpha_i)}$ is a direct sum of α_i copies of the group $Z(p^i)$, $i = 1, \ldots, n$; $\alpha_1, \ldots, \alpha_{n-1} \leq \aleph_0$ are cardinal numbers and $\alpha_n = \aleph_0$. Referring to [5], there exists a group G such that $p^\omega G = K$ and $\mathrm{E}(G) \,|\, K = L$, where L is the subring of the ring $\mathrm{E}(K)$ generated by the identity endomorphism. If now $H \leq K[p]$ and $H \cap p^{n-1}K$ is infinite having infinite index, then H is fully invariant in G, but $H \cap p^\sigma G$ is not commensurable with $(p^\sigma G)[p]$ for every ordinal $\sigma < \omega + n$.

(2) Suppose that G is the group from (1), where $n \geq 2$, $\alpha_1, \ldots, \alpha_n = \aleph_0$, $K_n[p] \leq H \leq K[p]$ and H is such that $H \cap K_i$ is infinite and has in each $K_i[p]$ infinite index; $i = 1, \ldots, n$. Then H is fully invariant in G and $H \cap p^{\omega+n-1}G =$

$(p^{\omega+n-1}G)[p]$, but H is not commensurable with $(p^\sigma G)[p]$ for every $\sigma < \tau = \omega + n$. □

It is pretty easy to show that for any ordinal α and any fully inertly socle-regular group G, the subgroup $p^\alpha G$ is always fully inertly socle-regular.

Theorem 2.12. *Let G be a group such that $G/p^\omega G$ is a direct sum of cyclic groups and $p^\omega G$ is infinite. Then G is fully inertly socle-regular if, and only if, $p^\omega G$ is fully inertly socle-regular.*

Proof. We have already noted early that G being fully inertly socle-regular implies that its subgroup $p^\alpha G$ is fully inertly socle-regular for any ordinal α, so it suffices to handle only the sufficiency.

In doing that, let H be an arbitrary infinite fully inert subgroup of G. Note however that if f is an arbitrary endomorphism of $p^\omega G$, then it follows from [14] that every endomorphism of $p^\omega G$ is induced from an endomorphism of G. So $H[p] \cap p^\omega G$ is a fully inert subgroup of $p^\omega G$. Note also that $H[p] \cap p^\omega G$ has to be infinite, which fact is fairly enough to end the proof. In fact, if for the otherwise, it follows that $H[p] = H' \oplus (H[p] \cap p^\omega G)$, where $|\,H'\,| \geq \aleph_0$. Since $G/p^\omega G$ is a direct sum of cyclic groups, it is possible to choose a composition φ of homomorphisms $G \to G/p^\omega G \to (p^\omega G)[p]$ such that $\varphi(H[p])$ is infinite and such φ can be considered as an endomorphism of the group G. So, we really obtain the infinity of the intersection $H[p] \cap p^\omega G$, as desired. □

Theorem 2.13. *If $G = A \oplus C$, where A is a direct sum of cyclic groups and C is a fully inertly socle-regular group, then G is too a fully inertly socle-regular group.*

Proof. The claim is pretty obvious, provided A is finite. So, assume that A is infinite and let $H \leq G[p]$ be an infinite fully inert subgroup. If $|\,H \cap B_n\,| \geq \aleph_0$, then by Lemma 2.5 (1) the theorem is proved. Suppose now that $|\,H \cap B_n\,| < \aleph_0$ for all $n \in \mathbb{N}$. Thus A is necessarily unbounded. Utilizing Lemma 2.2, H is commensurable with $(H \cap A) \oplus (H \cap C)$. Furthermore, one assumes that $H \cap A$ is commensurable with $(p^n A)[p]$ and $H \cap C$ is commensurable with $(p^m C)[p]$ (see Lemma 2.5 (3)). Since $|\,H \cap B_n\,| < \aleph_0$, H is commensurable with $(p^{max\{n,m\}}G)[p]$. Note that the case when $H \cap p^\sigma C$ is commensurable with $(p^\sigma C)[p]$, where $\sigma \geq \omega$ and $H \cap A$ is infinite, is impossible in view Lemma 2.5 (2), because in this case the group G is reduced and so the subgroup H is not fully inert in it. □

3. Concluding Discussion and Open Problems

In closing, in regard to our considerations alluded to above, we state a few questions of interest:

Problem 1. Are inertly fully transitive groups also fully inertly socle-regular?

It is worthwhile noticing that it was shown in [6] that fully transitive groups are themselves socle-regular.

Problem 2. Is a direct summand of a fully inertly socle-regular group also fully inertly socle-regular? Same question appears for inertly fully transitive groups.

Problem 3. Do there exist non-separable groups satisfying conditions (1) and (2) of Proposition 2.9?

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