Residual properties of virtual knot groups

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Braid group B_n on $n \geq 2$ strands is generated by $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ and is defined by relations

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{for } i = 1, 2, \ldots, n-2, \tag{1}$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for } |i-j| \geq 2. \tag{2}$$

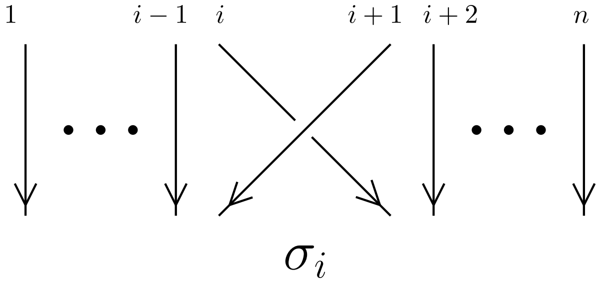

Figure: Geometric interpretation of σ_i

The Artin representation

$$\varphi_A : B_n \longrightarrow \mathrm{Aut}(F_n),$$

where $F_n = \langle x_1, x_2, \ldots, x_n \rangle$ is a free group, is defined by the rule

$$\varphi_A(\sigma_i) : \begin{cases} x_i \longmapsto x_i x_{i+1} x_i^{-1}, \\ x_{i+1} \longmapsto x_i, \end{cases}$$

Here and onward we point out only nontrivial actions on generators assuming that other generators are fixed.

Theorem [Artin]: $Ker(\varphi_A) = 1$.

Let $\mathcal{L}$ be the set of all links in $\mathbb{R}^3$.

A group $G(L)$ of a link $L \in \mathcal{L}$ is a group $\pi_1(\mathbb{R}^3 \setminus L)$.

Theorem [Artin]: If L is isotopic to $\hat{\beta}$, where $\beta \in B_n$, then

$$G(L) = \langle x_1, x_2, \ldots, x_n \parallel x_i = \varphi_A(\beta)(x_i), \quad i = 1, 2, \ldots, n \rangle.$$

The virtual braid group VB_n is presented by L. Kauffman (1996).
V. Vershinin constructed the more compact system of defining relations for VB_n.

VB_n is generated by the classical braid group $B_n = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$ and the permutation group $S_n = \langle \rho_1, \ldots, \rho_{n-1} \rangle$. Generators $\rho_i, i = 1, \ldots, n-1$, satisfy the following relations:

$$\rho_i^2 = 1 \qquad \text{for } i = 1, 2, \ldots, n-1, \tag{3}$$

$$\rho_i \rho_j = \rho_j \rho_i \qquad \text{for } |i-j| \geq 2, \tag{4}$$

$$\rho_i \rho_{i+1} \rho_i = \rho_{i+1} \rho_i \rho_{i+1} \qquad \text{for } i = 1, 2 \ldots, n-2. \tag{5}$$

Other defining relations of the group VB_n are mixed and they are as follows

$$\sigma_i \rho_j = \rho_j \sigma_i \qquad \text{for } |i-j| \geq 2, \tag{6}$$

$$\rho_i \rho_{i+1} \sigma_i = \sigma_{i+1} \rho_i \rho_{i+1} \qquad \text{for } i = 1, 2, \ldots, n-2. \tag{7}$$

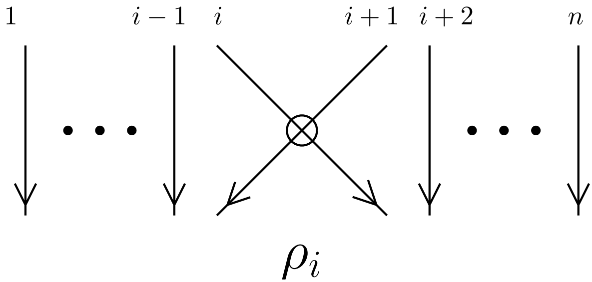

Figure: Geometric interpretation of ρ_i

- Construct a faithfull representation

$$\psi : VB_n \longrightarrow \mathrm{Aut}(H),$$

where H is a "good" group.

- Define a group of virtual links.

We consider the free product $F_{n,2n+1} = F_n * \mathbb{Z}^{2n+1}$, where F_n is a free group of the rank n generated by elements $x_1, x_2, \ldots, x_n$ and $\mathbb{Z}^{2n+1}$ is a free abelian group of the rank $2n+1$ freely generated by elements $u_1, u_2, \ldots, u_n$, $v_0, v_1, v_2, \ldots, v_n$.

Theorem 1 [V. B. – Yu. Mikhalchishina – M. Neshchadim, 2017].

The following mapping $\varphi_M : VB_n \longrightarrow \mathrm{Aut}(F_{n,2n+1})$ defined by the action on the generators:

$$\varphi_M(\sigma_i) : \begin{cases} x_i \longmapsto x_i x_{i+1}^{u_i} x_i^{-v_0 u_{i+1}}, \\ x_{i+1} \longmapsto x_i^{v_0}, \end{cases} \qquad \varphi_M(\sigma_i) : \begin{cases} u_i \longmapsto u_{i+1}, \\ u_{i+1} \longmapsto u_i, \end{cases}$$

$$\varphi_M(\sigma_i) : \begin{cases} v_i \longmapsto v_{i+1}, \\ v_{i+1} \longmapsto v_i, \end{cases}$$

$$\varphi_M(\rho_i) : \begin{cases} x_i \longmapsto x_{i+1}^{v_i^{-1}}, \\ x_{i+1} \longmapsto x_i^{v_{i+1}}, \end{cases} \qquad \varphi_M(\rho_i) : \begin{cases} u_i \longmapsto u_{i+1}, \\ u_{i+1} \longmapsto u_i, \end{cases}$$

$$\varphi_M(\rho_i) : \begin{cases} v_i \longmapsto v_{i+1}, \\ v_{i+1} \longmapsto v_i, \end{cases}$$

is provided a representation of VB_n into $\mathrm{Aut}(F_{n,2n+1})$, which generalizes all known representations.

The constructed representation φ_M is not an extension of the Artin representation.

It is turned out that the representation φ_M is equivalent to the simpler one which is an extension of the Artin representation.

Let $F_{n,n} = F_n * \mathbb{Z}^n$, where $F_n = \langle y_1, y_2, \ldots, y_n \rangle$ is the free group and $\mathbb{Z}^n = \langle v_1, v_2, \ldots, v_n \rangle$ is the free abelian group of the rank n.

Theorem 2 [V. B. – Yu. Mikhalchishina – M. Neshchadim, 2017].

The representation $\widetilde{\varphi}_M : VB_n \longrightarrow \mathrm{Aut}(F_{n,n})$ defined by the action on the generators

$$\widetilde{\varphi}_M(\sigma_i) : \begin{cases} y_i \longmapsto y_i y_{i+1} y_i^{-1}, \\ y_{i+1} \longmapsto y_i, \end{cases} \qquad \widetilde{\varphi}_M(\sigma_i) : \begin{cases} v_i \longmapsto v_{i+1}, \\ v_{i+1} \longmapsto v_i, \end{cases}$$

$$\widetilde{\varphi}_M(\rho_i) : \begin{cases} y_i \longmapsto y_{i+1}^{v_i^{-1}}, \\ y_{i+1} \longmapsto y_i^{v_{i+1}}, \end{cases} \qquad \widetilde{\varphi}_M(\rho_i) : \begin{cases} v_i \longmapsto v_{i+1}, \\ v_{i+1} \longmapsto v_i \end{cases}$$

is equivalent to the representation φ_M.

Assume that we have a representation $\psi : VB_n \longrightarrow \mathrm{Aut}(H)$ of the virtual braid group into the automorphism group of some group $H = \langle h_1, h_2, \ldots, h_m \parallel \mathcal{R} \rangle$, where $\mathcal{R}$ is the set of defining relations.

The following group is assigned to the virtual braid $\beta \in VB_n$:

$$G_\psi(\beta) = \langle h_1, h_2, \ldots, h_m \parallel \mathcal{R}, h_i = \psi(\beta)(h_i), \quad i = 1, 2, \ldots, m \rangle.$$

The group G_ψ is an invariant of virtual links if the group $G_\psi(\beta)$ is isomorphic to $G_\psi(\beta')$ for each braid β' such that the links $\widehat{\beta}$ and $\widehat{\beta'}$ are equivalent.

In the paper: J. S. Carter, D. Silver, S. Williams, *Invariants of links in thickened surfaces*, *Algebr. Geom. Topol.*, **14** (2014), no. 3, 1377–1394, suggested other approach to the definition of virtual link groups, which used interpretation of virtual link as a classical link in a thin surface.

This approach is used for the previously defined representation φ_M. Given $\beta \in VB_n$, the group of the braid β is the following group

$$G_M(\beta) = \langle x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_n, v_0, v_1, \ldots, v_n || [u_i, u_j] = [v_k, v_l] = [u_i, v_k] = 1,$$
$$x_i = \varphi_M(\beta)(x_i), \quad u_i = \varphi_M(\beta)(u_i), \quad v_i = \varphi_M(\beta)(v_i),$$
$$i, j = 1, 2, \ldots, n, \ k, l = 0, 1, \ldots, n\rangle.$$

Theorem 3 [V. B. – Yu. Mikhalchishina – M. Neshchadim, 2017].

Given $\beta \in VB_n$ and $\beta' \in VB_m$ the two virtual braids such that theirs closures define the same link L, then $G_M(\beta) \cong G_M(\beta')$.

Yu. Mikhalchishina (2017) defined the following three representations of the virtual braid group VB_n into $\mathrm{Aut}(F_{n+1})$, where $F_{n+1} = \langle y, x_1, x_2, \ldots, x_n \rangle$.

1. The representation $W_{1,r}$, $r > 0$ is defined by the action on the generators

$$W_{1,r}(\sigma_i) : \begin{cases} x_i \longmapsto x_i^r x_{i+1} x_i^{-r}, \\ x_{i+1} \longmapsto x_i, \end{cases} \qquad W_{1,r}(\rho_i) : \begin{cases} x_i \longmapsto x_{i+1}^{y^{-1}}, \\ x_{i+1} \longmapsto x_i^y. \end{cases}$$

2. The representation W_2 is defined by the action on the generators

$$W_2(\sigma_i) : \begin{cases} x_i \longmapsto x_i x_{i+1}^{-1} x_i, \\ x_{i+1} \longmapsto x_i, \end{cases} \qquad W_2(\rho_i) : \begin{cases} x_i \longmapsto x_{i+1}^{y^{-1}}, \\ x_{i+1} \longmapsto x_i^y. \end{cases}$$

3. The representation W_3 is defined by the action on the generators

$$W_3(\sigma_i) : \begin{cases} x_i \longmapsto x_i^2 x_{i+1}, \\ x_{i+1} \longmapsto x_{i+1}^{-1} x_i^{-1} x_{i+1}, \end{cases} \qquad W_3(\rho_i) : \begin{cases} x_i \longmapsto x_{i+1}^{y^{-1}}, \\ x_{i+1} \longmapsto x_i^{y}. \end{cases}$$

These representations extend Wada representations $w_{1,r}$, $r > 0$, w_2, w_3 of B_n into $\mathrm{Aut}(F_n)$, where $F_n = \langle x_1, x_2, \ldots, x_n \rangle$ is a free group of rank n.

Yu. A. Mikhalchishina for each virtual link defined three types of groups : $G_{1,r}(L)$, $G_2(L)$ and $G_3(L)$ that correspond to described representations. He proved that these groups are invariants of a virtual link L.

The Kishino knot is a non-trivial knot that is the connected sum of two trivial knots.

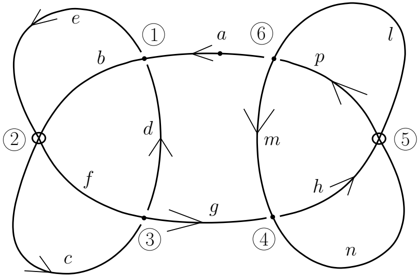

Figure: Kishino knot

Yu. Mikhalchishina proved that groups $G_{1,r}(Ki)$ and $G_2(Ki)$ cannot distinguish the Kishino knot Ki from the trivial one. She formulated the question: whether the group $G_3(Ki)$ is able to distinguish the Kishino knot from the trivial one or not?

Note that the group $G_3(U)$ of the trivial knot U is isomorphic to F_2.

Theorem 3 [V. B. – Yu. Mikhalchishina – M. Neshchadim, ArXiv, 2018].

The group $G = G_3(Ki)$ having generators a, b, c, d and the system of defining relations

$$d^{-1}b^{-d}c^{-2d^{-1}}b^{-d}c^{-2d^{-1}}aa^{-2d}d = a^{-1}b^{-d}c^{-2d^{-1}}a,$$

$$c^{-1}bc = b^{-d}c^{d^{-1}}b^{d},$$

$$c = b^{-d}c^{-2d^{-1}}b^{-d}c^{-2d^{-1}}aa^{-d}a^{-1}c^{2d^{-1}}b^{2d}.$$

is not isomorphic to the free group of rank 2.

A non trivial group G is called residually nilpotent if for any $1 \neq g \in G$ there is a nilpotent group N and a homomorphism $\varphi : G \longrightarrow N$ such that $\varphi(g) \neq 1$.

Note that if K is a non-trivial classical knot then its group $G(K)$ is not residually nilpotent since $[G(K), G(K)] = [[G(K), G(K)], G(K)]$.

For a group G define its transfinite lower central series

$$G = \gamma_1(G) \geq \gamma_2(G) \geq \ldots \geq \gamma_\omega(G) \geq \gamma_{\omega+1}(G) \geq \ldots,$$

where

$$\gamma_{\alpha+1}(G) = \langle [g_\alpha, g] \mid g_\alpha \in \gamma_\alpha(G), g \in G \rangle$$

and if α is a limit ordinal, then

$$\gamma_\alpha(G) = \bigcap_{\beta<\alpha} \gamma_\beta(G).$$

The minimal α such that $\gamma_\alpha(G) = 1$ is called the class of G.

If class of G is a finite number, then G is nilpotent. If class of G is ω, then G is residually nilpotent.

Example

1) (W. Magnus) If F is a non-abelian free group, then it is not nilpotent, but is residually nilpotent.
2) (A. I. Malcev, 1949) For any ordinal α there is a group of class α.

T. D. Cochran in 1990 formulated a question on residually nilpotent of link groups.

Theorem 3 [V. B. – R. Mikhailov, 2007].

1) If L is Whaithed link or Borromean rings, then $G(L)$ is residually nilpotent.
2) There exists a 2-component 2-bridge link whose group is not residually nilpotent.
3) Each link in $\mathbb{S}^3$ is a sublink of some link whose link group is residually nilpotent.

For a definition of Milnor's invariant of a link L one can use the quotients $G(L)/\gamma_m(G(L))$ of link group by some term of the lower central series.

Magnus constructed a representation of the free group $F_n = \langle x_1, x_2, \ldots, x_n \rangle$ into the ring of formal power series $\mathbb{Z}[[X_1,\, X_2,\, \ldots, X_n]]$ of noncommutative variables $X_1, X_2, \ldots, X_n$ defined by the action on generators:

$$x_i \mapsto 1 + X_i, \quad i = 1, 2, \ldots, n.$$

In that case inverse elements of generators go to the following formal power series

$$x_i^{-1} \mapsto 1 - X_i + X_i^2 - X_i^3 + \ldots, \quad i = 1, 2, \ldots, n.$$

The representation defined in that manner is faithful (i. e. its kernel is trivial).

Moreover, it remains being faithful if the ring $\mathbb{Z}[[X_1, X_2, \ldots, X_n]]$ is replaced by the quotient ring $\mathbb{Z}[[X_1, X_2, \ldots, X_n]]/\langle X_1^2, X_2^2, \ldots, X_n^2 \rangle$ by the two-sided ideal generated by elements $X_1^2, X_2^2, \ldots, X_n^2$.

Let

$$\mathcal{P} = \langle\, x_1, \ldots, x_n \,\|\, r_1, \ldots, r_m \,\rangle$$

be some finite presentation of the group G and $A_n = \mathbb{Q}[[X_1, \ldots, X_n]]$ is an algebra of formal power series of noncommutative variables $X_1, \ldots, X_n$ over the field of rational numbers. Define series f_j, $j = 1, \ldots, m$, in the algebra A_n by equalities

$$f_j = r_j(1 + X_1, \ldots, 1 + X_n) - 1, \quad j = 1, \ldots, m.$$

Proposition [V. B. – Yu. Mikhalchishina – M. Neshchadim, ArXiv 2018].

The quotient algebra $A_n / \langle f_1, \ldots, f_m \rangle$ is an invariant of the group G, i. e. it does not depend on the explicit presentation.

Let L be a virtual link and

$$G(L) = \langle\, x_1, \ldots, x_n \,|||\, r_1, \ldots, r_m \,\rangle$$

its group.

Corollary [V. B. – Yu. Mikhalchishina – M. Neshchadim, ArXiv 2018].

The quotient algebra $A_n / \langle f_1, \ldots, f_m \rangle$ is an invariant of L.

In this part I will formulate some results that we have found with Neha Nanda and M. Neshchadim.

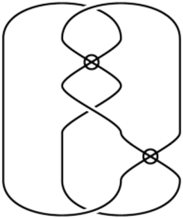

Figure: Knot K_1

The group $G(K_1)$ has the presentation

$$G(K_1) = \langle\, x,\, y \,\|\, [x^{-1},\, y,\, x^{-1},\, yx^{-1}] = 1 \,\rangle.$$

Proposition 1.

- Considering the quotient of $G(K_1)$ by 5-th term of the lower central series we can prove that K_1 is non-trivial:

$$G(K_1)/\gamma_4 G(K_1) \cong F_2/\gamma_4 F_2,$$

$$\gamma_4 G(K_1)/\gamma_5 G(K_1) \cong \mathbb{Z}^2, \quad G(K_1)/\gamma_5 G(K_1) \ncong F_2/\gamma_5 F_2.$$

- $G(K_1) = F_3 \leftthreetimes \mathbb{Z}$ is an extension of a free group F_3 of rank 3 by $\mathbb{Z} = \langle x \rangle$.
- $G(K_1)$ is residually nilpotent.

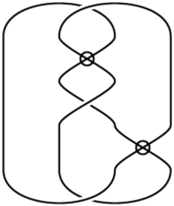

Figure: Knot K_2

The group $G(K_2)$ has the presentation

$$G(K_2) = \langle\, x,\, y \,\|\, [x^{y^{-1}xy^{-1}} x^{yx^{-1}y},\, x] = 1 \,\rangle.$$

Proposition 2.

- Considering the quotient of $G(K_2)$ by 5-th term of the lower central series we can prove that K_2 is non-trivial:

$$G(K_2)/\gamma_4 G(K_2) \cong F_2/\gamma_4 F_2,$$

$$\gamma_4 G(K_2)/\gamma_5 G(K_2) \cong \mathbb{Z}^2 \times \mathbb{Z}_4, \quad G(K_2)/\gamma_5 G(K_2) \ncong F_2/\gamma_5 F_2.$$

- $G(K_2) = F_5 \leftthreetimes \mathbb{Z}$ is an extension of a free group F_5 of rank 5 by $\mathbb{Z} = \langle x \rangle$.
- $\gamma_\omega G(K_2) \subseteq F_5'$, $\gamma_{\omega^2} G(K_2) = 1$.

Question.

Is it true that $\gamma_\alpha G(K_2) \neq 1$ for all $\alpha < \omega^2$?

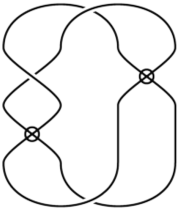

Figure: Knot K_3

The group $G(K_3)$ has the presentation

$$G(K_3) = \langle\, x,\, y \,\|\, x = x^{-y^2} x^{-1} x^{-y^{-2}} x x^{y^{-2}} x x^{y^2} \,\rangle.$$

Proposition 3.

- Considering the quotient of $G(K_3)$ by 5-th term of the lower central series we can prove that K_2 is non-trivial:

$$G(K_3)/\gamma_4 G(K_3) \cong F_2/\gamma_4 F_2,$$

$$\gamma_4 G(K_3)/\gamma_5 G(K_3) \cong \mathbb{Z}^2 \times \mathbb{Z}_4, \quad G(K_3)/\gamma_5 G(K_3) \ncong F_2/\gamma_5 F_2.$$

- $G(K_3) = H_3 \leftthreetimes \mathbb{Z}$, where $\mathbb{Z} = \langle y \rangle$, $x_k^y = x_{k+1}$, $k \in \mathbb{Z}$,

$$H_3 = \ldots \underset{B_k}{*} A_k \underset{B_{k+1}}{*} A_{k+1} \underset{B_{k+2}}{*} A_{k+2} \underset{B_{k+3}}{*} \ldots,$$

and $A_k = \langle\, x_k,\, x_{k+1},\, x_{k+2},\, x_{k+3},\, x_{k+4},\ \|\, [x_k, x_{k+2}] = [x_{k+2}^{-1}, x_{k+4}^{-1}]\,\rangle$,
$B_k = \langle\, x_k,\, x_{k+1},\, x_{k+2},\, x_{k+3}\,\rangle \cong F_4$.

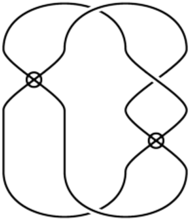

Figure: Knot K_4

The group $G(K_4)$ has the presentation

$$G(K_4) = \langle\, x_1,\, x_2,\, y \,\|\, x_1 x_2^{-1} = [x_2^{-y^{-2}}, x_1],\ x_2^{-1} x_1 = [x_1^{-y^2}, x_2^{-1}] \,\rangle.$$

This group has two defining relations.

Proposition 4.

$$G(K_4)/\gamma_4 G(K_4) \cong F_2/\gamma_4 F_2.$$

Questions.

- Is it possible to prove that K_4 is non-trivial, considering the quotient $G(K_4)/[\gamma_2 G(K_4), \gamma_2 G(K_4)]$?
- Is it true that $G(K_4)$ is a parafree group?

Recall that a group is said to be parafree if its quotients by the terms of its lower central series are the same as those of a free group and if it is residually nilpotent.

- Let K be a virtual knot and its group $G_M(K)$ is non isomorphic to the group of trivial knot. Is it true that if $\gamma_2 G_M(K) \neq \gamma_3 G_M(K)$, then K is not equivalent to a classical knot?
- Construct a non-trivial virtual knot K such that $G_3(K) \cong F_2$.

Thank you!