Volume of a compact hyperbolic tetrahedron in terms of its edge matrix

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Euclidean tetrahedron

The calculation of the volume of a polyhedron in 3-dimensional space E^3, H^3, or S^3 is a very old and difficult problem. The first known result belongs to **Tartaglia (1499-1557)** who had described an algorithm for calculating the height of a tetrahedron with some concrete lengths of its edges. The formula which expresses the volume of an Euclidean tetrahedron in terms of its edge lengths was given by Euler. The multidimensional analogue of this result is known as the Cayley–Menger determinant. More precisely, let T be an Euclidean tetrahedron with edge lengths d_{ij}, $1 \leq i < j \leq 4$. Then $V = Vol(T)$ is given by

$$288\,V^2 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & d_{12}^2 & d_{13}^2 & d_{14}^2 \\ 1 & d_{21}^2 & 0 & d_{23}^2 & d_{24}^2 \\ 1 & d_{31}^2 & d_{32}^2 & 0 & d_{34}^2 \\ 1 & d_{41}^2 & d_{42}^2 & d_{43}^2 & 0 \end{vmatrix}.$$

Here V is a root of quadratic equation whose coefficients are integer polynomials in d_{ij}, $1 \leq i < j \leq 4$.

Introduction

The problem of calculating volumes of polyhedra remains relevant nowadays. This is partly due to the fact that the volume of a fundamental polyhedron is main geometrical invariant for a 3-dimensional manifold. Since Mostov rigidity theorem it is a topological invariant also.

Every 3-manifold can be presented by a fundamental polyhedron. That means we can pair-wise identify the faces of some polyhedron to construct a 3-manifold. Thus the volume of 3-manifold is the volume of its fundamental polyhedron.

Introduction

Theorem (Thurston, Jørgensen)

The volumes of 3-dimensional hyperbolic manifolds form a closed non-discrete set on the real line. This set is well ordered. There are only finitely many manifolds with a given volume.

J. Weeks (1985), S. Matveev and A. Fomenko (1988) constructed a closed hyperbolic 3-manifold obtained by $(5, 2)$ and $(5, 1)$ Dehn surgeries on the Whitehead link. Its volume is $0,9427\ldots$ In 2009, D. Gabai, R. Meyerhoff, P. Milley showed that it has the smallest volume of any closed orientable hyperbolic 3-manifold.

Second smallest manifold was given by W. Thurston (1980) by $(5, 1)$ Dehn surgery on the figure-eight knot. Its volume is $0,98\ldots$

Third known smallest manifold is Meyerhof–Neumann manifold (1992). Its volume is $1,01\ldots$

Some motivation to find exact volume formulas

Rational volume conjecture. Let P be a spherical polyhedron whose dihedral angles are in $\pi \cdot \mathbb{Q}$. Then $Vol\,(P) \in \pi^2 \cdot \mathbb{Q}$.

- Examples

1. Let L, be a spherical Lambert cube with dihedral angles $\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{3\pi}{4}$. Then
$$Vol\,L\left(\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{3\pi}{4}\right) = \frac{31}{576}\,\pi^2.$$
2. Let P be a Coxeter polyhedron in S^3 (that is all dihedral angles of P are $\frac{\pi}{n}$ for some $n \in \mathbb{N}$). Then the Coxeter group $\Delta(P)$ generated by reflections in faces of P is finite and
$$Vol\,(P) = \frac{Vol\,(S^3)}{|\Delta(P)|} = \frac{2\,\pi^2}{|\Delta(P)|} \in \pi^2 \cdot \mathbb{Q}\,.$$
3. A. Kolpakov (2015) constructed another infinite series of spherical polyhedra with diedral angles in $\pi \cdot \mathbb{Q}$ and volumes in $\pi^2 \cdot \mathbb{Q}$

Some motivation to find exact volume formulas

W. Thurston suggest to show that volumes of hyperbolic 3-manifolds are not all rationally related (problem #23 (1982), proposed before by J. Milnor, still open).

First examples of hyperbolic 3-manifolds

1914 Gieseking found first example of hyperbolic manifold (non-compact, non-orientable)

1929 Klein wrote in his book «Non-Euclidean Geometry» that examples of compact hyperbolic 3-manifolds are unknown

1931 Löbell presented the example of compact orientable hyperbolic 3-manifold

1933 Weber and Seifert constructed compact orientable «dodecahedral hyperbolic space»

Examples of geom. structures on knots complements in $\mathbb{S}^3$

1975 R. Riley found first examples of hyperbolic structures on seven *excellent* knots and links in $\mathbb{S}^3$.

1977 W. Thurston showed that a complement of any prime knot admits a hyperbolic structure if this knot is not toric or satellite one.

1980 W. Thurston constructed a hyperbolic 3-manifold homeomorphic to the complement of knot 4_1 in $\mathbb{S}^3$ by gluing faces of two regular ideal tetrahedra. This manifold has a complete hyperbolic structure.

1982 J. Minkus suggested a general topological construction for the orbifold whose singular set is a two-bridge knot in $\mathbb{S}^3$.

2004 H. Hilden, J. Montesinos, D. Tejada, M. Toro considered more general topological construction known as *butterfly*.

1998/2006 A. Mednykh, A. Rasskazov found a geometrical realisation of the Minkus construction in $\mathbb{H}^3, \mathbb{S}^3, \mathbb{E}^3$.

2009 E. Molnár, J. Szirmai, A. Vesnin realised the figure-eight knot cone-manifold in the five exotic Thurston's geometries.

Upper half-space model of hyperbolic 3-space

Denote by $\mathbb{H}^3$ a 3-dim *hyperbolic space* (*Lobachevsky–Boljai–Gauss space*).
$\mathbb{H}^3$ can be modelled in $\mathbb{R}^3_+ = \{(x, y, t) \ : \ x, y, t \in \mathbb{R}, t > 0\}$ with metric s given by expression $ds^2 = \dfrac{dx^2 + dy^2 + dt^2}{t^2}$.
The boundary $\partial\mathbb{H}^3 = \{(x, y, 0) \ : \ x, y \in \mathbb{R}\}$ caled *absolute* and consist of points at infinity.

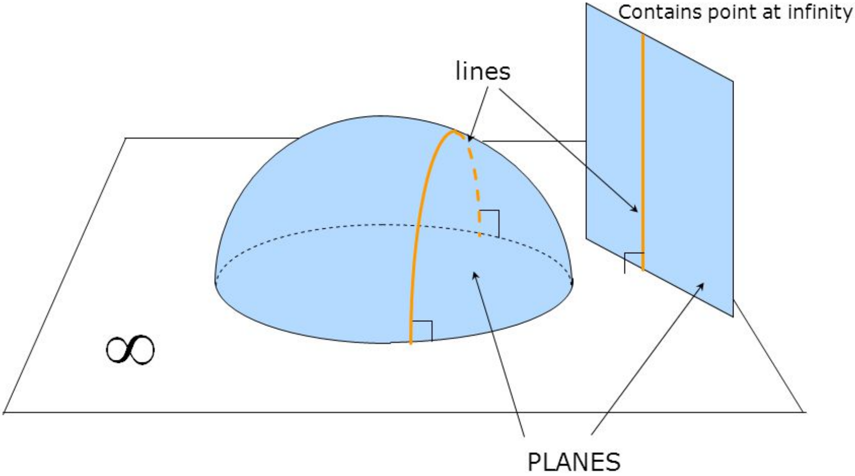
Isometry group $\mathrm{Isom}(\mathbb{H}^3)$ is a group of all actions on $\mathbb{H}^3$ preserving the metric s. Denote by $\mathrm{Isom}^+(\mathbb{H}^3)$ the group of orientation preserving isometries.

$\mathrm{Isom}^+(\mathbb{H}^3) \cong \mathbf{PSL}(2, \mathbb{C})$ (Pozitive Special Lorentz group). An element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{PSL}(2, \mathbb{C})$ acts on $\mathbb{H}^3$ by the rule

$$g \ : \ (z, t) \mapsto \left(\frac{(az + b)\overline{(cz + d)} + a\bar{c}t^2}{|cz + d|^2 + |c|^2 \, t^2}, \frac{t}{|cz + d|^2 + |c|^2 \, t^2} \right),$$

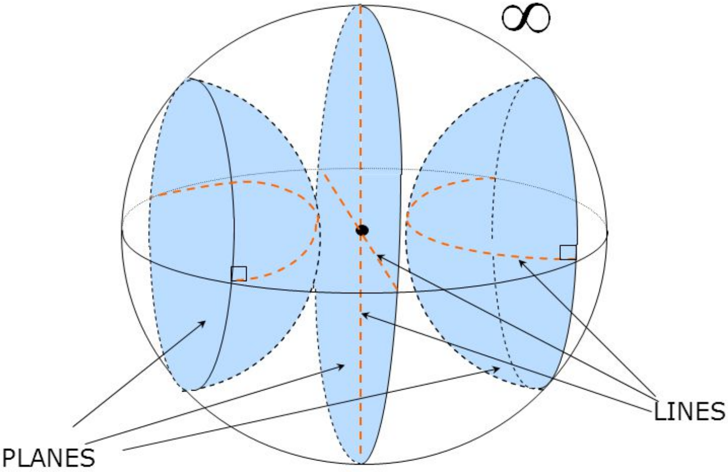
where $z = x + i\, y$.

Geodesic lines and planes in half-space model of $\mathbb{H}^3$



$\mathrm{Isom}(\mathbb{H}^3)$ is generated by reflections with respect to geodesic planes.

Geodesic lines and planes in ball model of $\mathbb{H}^3$



$\mathrm{Isom}(\mathbb{H}^3)$ is generated by reflections with respect to geodesic planes.

Constructing manifolds from polyhedra

Consider a *right-angled polyhedron* P (i.e. all the dihedral and planar angles of P are $\pi/2$). In Euclidean space we can take a cube. In the spherical space there is a right-angled tetrahedron ($1/8$ part of $\mathbb{S}^3$). In the hyperbolic space there are infinitely many right-angled polyhedra.

The class of polyhedra that can be realised in hyperbolic geometry with right angles is referred as *Pogorelov polyhedra*.

It follows from Andreev theorem (1968) that any polyhedron which has no triangle and quadrilateral faces and such that any its vertex is of valency 3, can be realised as right-angled polyhedron in $\mathbb{H}^3$.

Example

- n-gonal Löbell prism $R(n), n > 4$;
- all combinatorial fullerenes (including known in chemistry $C_{60}, C_{70}, C_{78}, C_{84}, C_{200}$ etc.)

$R(5) = C_{12} =$ *dodecahedron*.

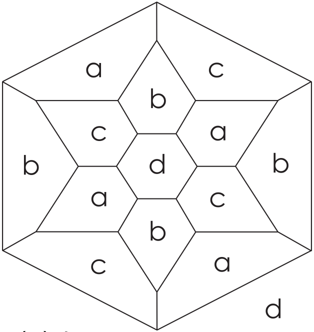
Constructing manifolds from polyhedra

To construct hyperbolic manifolds one can follow the algorithm:

1. take a compact right-angled hyperbolic polyhedron P;
2. set a regular colouring of the faces of P (the incidental faces should have different colours; the number of colours will be from 3 to 7);
3. pairwise identify the faces of same colour of several copies of P.

This approach was originally used by Löbell for $R(6)$ (1931) to construct the first example of a compact hyperbolic manifold. M. Takahashi (1985) do this for regular right-angled dodecahedron (or $R(5)$). A. Vesnin (1987) generalised this construction for any compact hyperbolic right-angled polyhedron P. All the manifolds constructed by colourings in 4 colours are orientable. If one use $5, 6$ or 7 colours then non-orientable hyperbolic manifolds can be produced.

Löbell construction of the first compact hyperbolic manifold



Consider a Löbell prism $R(6)$ having 12 pentagonal lateral faces. Let us set the colouring with 4 colours a, b, c, d. We take 8 copies of this coloured $R(6)$. Then identify faces of this 8 copies using the rule:

$$a : (15)(26)(37)(48)$$

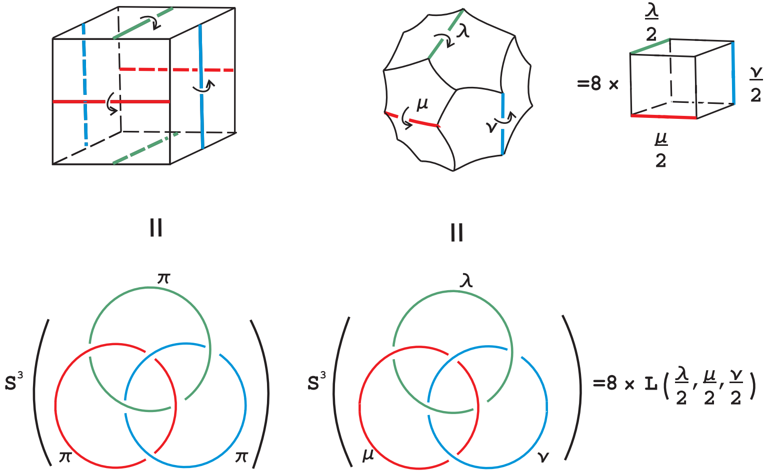
$$b : (16)(25)(38)(47)$$

$$c : (17)(28)(35)(46)$$

$$d : (18)(27)(36)(45)$$

- Borromean Rings cone-manifold and Lambert cube

This construction done by W. Thurston, D. Sullivan and J.M. Montesinos.



From the above consideration we get

$$\mathrm{Vol}\, B(\lambda,\mu,\nu) = 8 \cdot \mathrm{Vol}\, L\left(\frac{\lambda}{2}, \frac{\mu}{2}, \frac{\nu}{2}\right).$$

Recall that $B(\lambda,\mu,\nu)$ is

i) hyperbolic iff $0 < \lambda,\mu,\nu < \pi$ **(E.M. Andreev)**

ii) Euclidean iff $\lambda = \mu = \nu = \pi$

iii) spherical iff $\pi < \lambda,\mu,\nu < 3\pi, \quad \lambda,\mu,\nu \neq 2\pi$
(R. Diaz, D. Derevnin, A. Mednykh)

From polyhedra to knots and links

- Volume calculation for $L(\alpha, \beta, \gamma)$. The main idea.

0. Existence

$$L(\alpha, \beta, \gamma) : \begin{cases} 0 < \alpha, \beta, \gamma < \pi/2, & \mathbb{H}^3 \\ \alpha = \beta = \gamma = \pi/2, & \mathbb{E}^3 \\ \pi/2 < \alpha, \beta, \gamma < \pi, & \mathbb{S}^3. \end{cases}$$

1. Schläfli formula for $V = Vol\, L(\alpha, \beta, \gamma)$

$$k\, \mathrm{d}V = \frac{1}{2}(\ell_\alpha d\alpha + \ell_\beta d\beta + \ell_\gamma d\gamma), \quad k = \pm 1, 0\ (curvature)$$

In particular in hyperbolic case:

$$\begin{cases} \dfrac{\partial V}{\partial \alpha} = -\dfrac{\ell_\alpha}{2}, \quad \dfrac{\partial V}{\partial \beta} = -\dfrac{\ell_\beta}{2}, \quad \dfrac{\partial V}{\partial \gamma} = -\dfrac{\ell_\gamma}{2} & (*) \\ \mathrm{Vol}\, L\left(\dfrac{\pi}{2}, \dfrac{\pi}{2}, \dfrac{\pi}{2}\right) = 0. & (**) \end{cases}$$

2. Relations between lengths and angles

(i) Tangent Rule

$$\frac{\tan\alpha}{\tanh\ell_\alpha} = \frac{\tan\beta}{\tanh\ell_\beta} = \frac{\tan\gamma}{\tanh\ell_\gamma} =: T \quad (\textbf{Kellerhals})$$

(ii) Sine-Cosine Rule (3 different cases)

$$\frac{\sin\alpha}{\sinh\ell_\alpha}\frac{\sin\beta}{\sinh\ell_\beta}\frac{\cos\gamma}{\cosh\ell_\gamma} = 1 \quad (\textbf{Derevnin}, \textbf{Mednykh})$$

(iii) Tangent Rule

$$\frac{T^2 - A^2}{1 + A^2}\frac{T^2 - B^2}{1 + B^2}\frac{T^2 - C^2}{1 + C^2}\frac{1}{T^2} = 1, \quad (\textbf{Hilden}, \textbf{Lozano}, \textbf{Montesinos})$$

where $A = \tan\alpha, B = \tan\beta, C = \tan\gamma$. Equivalently,

$$(T^2 + 1)(T^4 - (A^2 + B^2 + C^2 + 1)T^2 + A^2B^2C^2) = 0.$$

Remark. (ii) $\Rightarrow$(i) and (i) & (ii) $\Rightarrow$ (iii).

3. Integral formula for volume

Hyperbolic volume of $L(\alpha,\beta,\gamma)$ is given by

$$W = \frac{1}{4}\int_{T}^{\infty} \log\left(\frac{t^2-A^2}{1+A^2}\,\frac{t^2-B^2}{1+B^2}\,\frac{t^2-C^2}{1+C^2}\,\frac{1}{t^2}\right)\frac{\mathrm{d}t}{1+t^2},$$

where T is a positive root of the integrant equation (iii).

Proof. By direct calculation and Tangent Rule (i) we have:

$$\frac{\partial W}{\partial \alpha} = \frac{\partial W}{\partial A}\frac{\partial A}{\partial \alpha} = -\frac{1}{2}\arctan\frac{A}{T} = -\frac{\ell_\alpha}{2}.$$

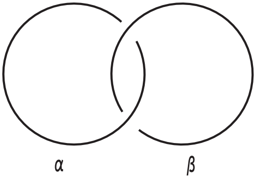
In a similar way

$$\frac{\partial W}{\partial \beta} = -\frac{\ell_\beta}{2} \quad \text{and} \quad \frac{\partial \mathrm{W}}{\partial \gamma} = -\frac{\ell_\gamma}{2}.$$

By convergence of the integral $W(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2}) = 0$. Hence, $W = V = Vol\, L(\alpha,\beta,\gamma)$.

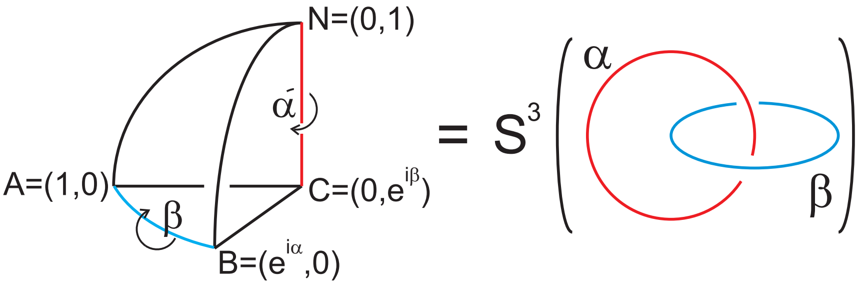
- The Hopf link

The Hopf link 2_1^2 is the simplest two component link.



The fundamental group $\pi_1(\mathbb{S}^3 \backslash 2_1^2) = \mathbb{Z}^2$ is a free Abelian group of rank 2. It makes us sure that **any finite covering of $\mathbb{S}^3 \backslash 2_1^2$ is homeomorphic to $\mathbb{S}^3 \backslash 2_1^2$ again**.
The orbifold $2_1^2(\pi, \pi)$ arises as a factor space by $\mathbb{Z}_2$-action on the projective space $\mathbb{P}^3$. That is, $\mathbb{P}^3$ **is a two-fold covering of the sphere $\mathbb{S}^3$ branched over the Hopf link**. It turns that **the sphere $\mathbb{S}^3$ is a two-fold unbranched covering of the projective space $\mathbb{P}^3$**.

- The Hopf link (Construction by Abr. and Mednykh)



Fundamental tetrahedron

$$\mathcal{T}(\alpha,\beta)=\mathcal{T}\left(\alpha,\beta,\tfrac{\pi}{2},\tfrac{\pi}{2},\tfrac{\pi}{2},\tfrac{\pi}{2}\right)\in\mathbb{S}^3\subset\mathbb{R}^4=\mathbb{C}\times\mathbb{C}$$

for the cone-manifold $2_1^2(\alpha,\beta)$.

Relations between lengths and angles: $\ell_\alpha=\beta,\ \ell_\beta=\alpha.$

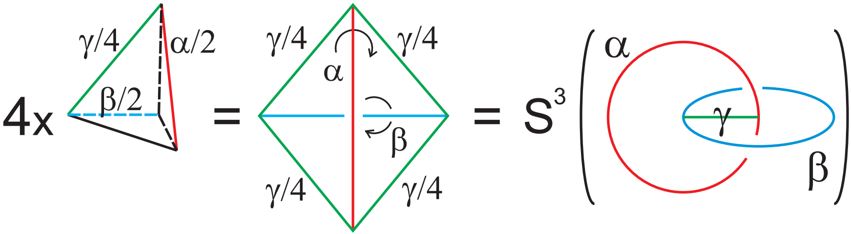
Theorem (Abr., Mednykh)

The Hopf link cone-manifold $2_1^2(\alpha, \beta)$ *is spherical for all positive* α *and* β. *The spherical volume is given by the formula*

$$\mathrm{Vol}\, 2_1^2(\alpha, \beta) = \frac{\alpha\,\beta}{2}.$$

Proof. Let $0 < \alpha, \beta \leqslant \pi$. Consider a spherical tetrahedron $\mathcal{T}(\alpha, \beta)$ with dihedral angles α and β prescribed to the opposite edges and with right angles prescribed to the remained ones. To obtain a cone-manifold $2_1^2(\alpha, \beta)$ we identify the faces of tetrahedron by α- and β-rotations in the respective edges. Hence, $2_1^2(\alpha, \beta)$ is spherical and $\mathrm{Vol}\, 2_1^2(\alpha, \beta) = \mathrm{Vol}\, \mathcal{T}(\alpha,\, \beta) = \frac{\alpha\,\beta}{2}$. We note that $\mathcal{T}(\alpha,\, \beta)$ is a union of n^2 tetrahedra $\mathcal{T}(\frac{\alpha}{n},\, \frac{\beta}{n})$. Hence, for large positive α and β we also obtain $\mathrm{Vol}\, 2_1^2(\alpha, \beta) = n^2 \cdot \mathrm{Vol}\, \mathcal{T}(\frac{\alpha}{n},\, \frac{\beta}{n}) = \frac{\alpha\,\beta}{2}$.

- The Hopf link with bridge (Construction by Abr. and Mednykh)



Fundamental tetrahedron $\mathcal{T}\left(\alpha,\beta,\frac{\gamma}{4},\frac{\gamma}{4},\frac{\gamma}{4},\frac{\gamma}{4}\right)$

for the Hopf link with bridge cone-manifold $\mathcal{H}(\alpha,\beta;\gamma)$.

Geometry of two bridge knots and links

- The Hopf link with bridge

Relations between lengths and angles:

Tangent Rule (Abr., Mednykh)

$$\tan\frac{\alpha}{2}\,\tanh\frac{\ell_\alpha}{2} = \frac{\tanh\ell_\gamma}{\tan\frac{\gamma}{4}} = \tan\frac{\beta}{2}\,\tanh\frac{\ell_\beta}{2}$$

Sine-Cosine Rule (Abr., Mednykh)

$$\frac{\cos\frac{\gamma}{4}}{\cosh\ell_\gamma} = \frac{\sin\frac{\alpha}{2}}{\cosh\frac{\ell_\alpha}{2}} \cdot \frac{\sin\frac{\beta}{2}}{\cosh\frac{\ell_\beta}{2}}$$

Given α, β, γ these theorems are sufficient to determine $\ell_\alpha, \ell_\beta, \ell_\gamma$. This allows us to use Schläfli equation: we are able to solve the system of PDE's to get the volume formula.

- The Hopf link with bridge

Theorem (Abr., Mednykh)

The Hopf link with bridge cone manifold $\mathcal{H}(\alpha, \beta; \gamma)$ is hyperbolic for any $\alpha, \beta \in (0, \pi)$ if and only if

$$\begin{cases} \gamma > 2(\pi - \alpha) \\ \gamma > 2(\pi - \beta) \\ \gamma < 2\pi \end{cases}$$

The hyperbolic volume is given by the formula

$\mathrm{Vol}\, \mathcal{H}(\alpha, \beta; \gamma) = i \cdot S\left(\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{4}\right)$, *where* $S\left(\frac{\pi}{2} - x, y, \frac{\pi}{2} - z\right) =$

$$\tilde{S}(x, y, z) = \sum_{m=1}^{\infty} \left(\frac{D - \sin x \sin z}{D + \sin x \sin z}\right)^m \cdot \frac{\cos 2mx - \cos 2my + \cos 2mz - 1}{m^2} - x^2 + y^2 - z^2$$

is the Schläfli function.

Volume of a hyperbolic tetrahedron

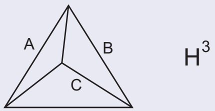
It is difficult problem to find the exact volume formulas for hyperbolic polyhedra of prescribed combinatorial type. It was done for hyperbolic tetrahedron of general type, but even for general hyperbolic octahedron it is an open problem.

Nevertheless, if we know that a polyhedron has a symmetry, then the volume calculation is essentially simplified. Firstly this effect was shown by Lobachevsky. He found the volume of an ideal tetrahedron, which is symmetric by definition.

Hyperbolic orthoscheme

The following theorem is the Coxeter's version of the Lobachevsky result.

Theorem (Coxeter, 1936)



The volume of a hyperbolic orthoscheme with essential dihedral angles A, B, C is given by the formula $V = \frac{i}{4} S(A, B, C)$, *where*

$$S(\frac{\pi}{2} - x, y, \frac{\pi}{2} - z) = \widehat{S}(x, y, z) =$$

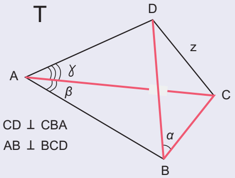
$$\sum_{m=1}^{\infty} \left(\frac{D - \sin x \sin z}{D + \sin x \sin z} \right)^m \frac{\cos 2mx - \cos 2my + \cos 2mz - 1}{m^2} - x^2 + y^2 - z^2$$

and $D \equiv \sqrt{\cos^2 x \cos^2 z - \cos^2 y}$.

The volume of a biorthogonal tetrahedron (orthoscheme) was calculated by Lobachevsky and Bolyai in H^3 and by Schläfli in S^3.

Theorem (J. Bolyai)

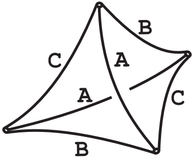
The volume of hyperbolic orthoscheme T is given by the formula



$$Vol\,(T) = \frac{\tan\gamma}{2\tan\beta}\int\limits_0^z \frac{z\sinh z\,dz}{\left(\frac{\cosh^2 z}{\cos^2\alpha}-1\right)\sqrt{\frac{\cosh^2 z}{\cos^2\gamma}-1}}.$$

Ideal tetrahedron

Consider an ideal hyperbolic tetrahedron T with all vertices at the infinity



Opposite dihedral angles of an ideal tetrahedron are equal to each other and $A+B+C=\pi$.

Theorem (J. Milnor, 1982)

$$Vol(T)=\Lambda(A)+\Lambda(B)+\Lambda(C),\ \textit{where}\ \Lambda(x)=-\int_0^x \log|2\sin t|\,dt$$

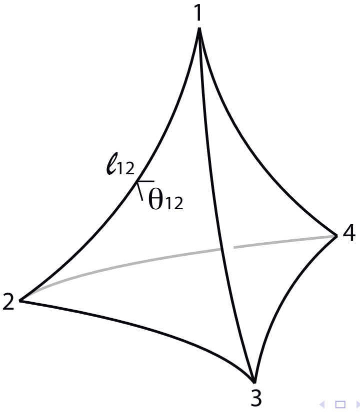
is the Lobachevsky function.

More complicated case with only one vertex at the infinity was investigated by **E. B. Vinberg (1993)**.

Definition

A *compact hyperbolic tetrahedron* T is a convex hull of four points in the hyperbolic space $\mathbb{H}^3$.

Let us denote the vertices of T by numbers $1, 2, 3$ and 4. Then denote by ℓ_{ij} the length of the edge connecting i-th and j-th vertices. We put θ_{ij} for the dihedral angle along the corresponding edge.



Gram matrix $G(T)$ of a tetrahedron T

Definition

A *Gram matrix* $G(T)$ of tetrahedron T is defined as
$G(T) = \langle -\cos\theta_{ij} \rangle_{i,j=1,2,3,4} =$

$$\begin{pmatrix} 1 & -\cos\theta_{12} & -\cos\theta_{13} & -\cos\theta_{14} \\ -\cos\theta_{12} & 1 & -\cos\theta_{23} & -\cos\theta_{24} \\ -\cos\theta_{13} & -\cos\theta_{23} & 1 & -\cos\theta_{34} \\ -\cos\theta_{14} & -\cos\theta_{24} & -\cos\theta_{34} & 1 \end{pmatrix},$$

we assume here that $-\cos\theta_{ii} = 1$.

Edge matrix $E(T)$ of a tetrahedron T

Definition

An *edge matrix* $E(T)$ of a hyperbolic tetrahedron T is defined as

$$E(T) = \langle \cosh \ell_{ij} \rangle_{i,j=1,2,3,4} = \begin{pmatrix} 1 & \cosh \ell_{12} & \cosh \ell_{13} & \cosh \ell_{14} \\ \cosh \ell_{12} & 1 & \cosh \ell_{23} & \cosh \ell_{24} \\ \cosh \ell_{13} & \cosh \ell_{23} & 1 & \cosh \ell_{34} \\ \cosh \ell_{14} & \cosh \ell_{24} & \cosh \ell_{34} & 1 \end{pmatrix},$$

where $\ell_{ii} = 0$ and $\cosh \ell_{ii} = 1$.

It is well known that T is uniquely defined up to isometry either by the set of its dihedral angles or the set of its edge lengths. So, either a Gram matrix $G(T)$ or an edge matrix $E(T)$ contains all the information about a hyperbolic tetrahedron T. This is unlikely to Euclidean case, where only an edge matrix defines a tetrahedron up to isometry.

Hyperbolic tetrahedron

Despite of the above mentioned partial results, a volume formula for arbitrary hyperbolic tetrahedron has been unknown until recently. The general algorithm for obtaining such a formula was indicated by **W.–Y. Hsiang (1988)** and the complete solution of the problem was given by **Yu. Cho and H. Kim (1999)**, **J. Murakami, M. Yano (2001)** and **A. Ushijima (2002)**.

In these papers the volume of tetrahedron is expressed as an analytic formula involving 16 Dilogarithm of Lobachevsky functions whose arguments depend on the dihedral angles of the tetrahedron and on some additional parameter which is a root of some complicated quadratic equation with complex coefficients.
A geometrical meaning of the obtained formula was recognized by **G. Leibon** from the point of view of the *Regge symmetry*. An excellent exposition of these ideas and a complete geometric proof of the volume formula was given by **Y. Mohanty (2003)**.

Murakami-Yano's formula

Recall that the Dilogarithm function is defined by

$$\mathrm{Li}_2(x) = -\int_0^x \frac{\log(1-t)}{t}\,dt.$$

We set $\ell(z) = \mathrm{Li}_2(e^{iz})$ and note that $\Im(\ell(z)) = 2\,\Lambda(\frac{z}{2})$.

Theorem (J. Murakami, M. Yano, 2001)

$$Vol(T) = \tfrac{1}{2}\Im(U(z_1, T) - U(z_2, T)), \ \ where$$

$$U(z,T) = \frac{1}{2}(\ell(z) + \ell(A+B+D+E+z)$$
$$+\ell(A+C+D+F+z) + \ell(B+C+E+F+z)$$
$$-\ell(\pi+A+B+C+z) - \ell(\pi+A+E+F+z)$$
$$-\ell(\pi+B+D+F+z) - \ell(\pi+C+D+E+z)).$$

Derevnin-Mednykh's formula

We suggest the following version of the integral formula for the volume. Let $T = T(A, B, C, D, E, F)$ be a hyperbolic tetrahedron with dihedral angles A, B, C, D, E, F. We set

$V_1 = A + B + C$, $V_2 = A + E + F$, $V_3 = B + D + F$, $V_4 = C + D + E$ (for vertices)

$H_1 = A + B + D + E$, $H_2 = A + C + D + F$, $H_3 = B + C + E + F$, $H_4 = 0$ (for Hamiltonian cycles).

Theorem (D. Derevnin, A. Mednykh, 2005)

The volume of a hyperbolic tetrahedron is given by the formula

$$Vol\,(T) = -\frac{1}{4}\int\limits_{z_1}^{z_2} \log \prod_{i=1}^{4} \frac{\cos\frac{V_i+z}{2}}{\sin\frac{H_i+z}{2}}\,dz,$$

where z_1 and z_2 are appropriate roots of the integrand.

More precisely, the roots in the previous theorem are given by the formulas

$$z_1 = \arctan \frac{K_2}{K_1} - \arctan \frac{K_4}{K_3}, \;\; z_2 = \arctan \frac{K_2}{K_1} + \arctan \frac{K_4}{K_3}$$

and

$$K_1 = -\sum_{i=1}^{4} (\cos(S - H_i) + \cos(S - V_i)),$$

$$K_2 = \sum_{i=1}^{4} (\sin(S - H_i) + \sin(S - V_i)),$$

$$K_3 = 2(\sin A \sin D + \sin B \sin E + \sin C \sin F),$$

$$K_4 = \sqrt{K_1^2 + K_2^2 - K_3^2}, \qquad S = A + B + C + D + E + F.$$

Sforza's formula

Theorem (G. Sforza, 1906)

Let T be a compact hyperbolic tetrahedron with **Gram matrix** *G. We assume that all the dihedral angles are fixed exept θ_{34} which is formal variable. Then the volume $V = V(T)$ is given by the formula*

$$Vol\,(T) = \frac{1}{4}\int\limits_{t_0}^{\theta_{34}} \log \frac{c_{34}(t) - \sqrt{-\det\ G(t)}\sin t}{c_{34}(t) + \sqrt{-\det\ G(t)}\sin t}dt,$$

where t_0 is a suitable root of the equation $\det G(t) = 0$, c_{34} is $(3,4)$-cofactor of the matrix G, and $c_{34}(t)$, $G(t)$ are functions in one variable θ_{34} denoted by t.

New and simple proof of the Sforza's formula was given by Abr. and A. Mednykh (2014). Proof

Known facts about Gram matrix of a hyperbolic tatrahedron

Theorem (A. Ushijima, 2003; A. Mednykh, M. Pashkevich, 2006)

Let T be a compact hyperbolic tetrahedron with **Gram matrix** G. *Then*

$$\begin{array}{ll} \text{(i)} & \det G < 0; \\ \text{(ii)} & c_{ii} > 0,\ i \in \{1,2,3,4\}; \\ \text{(iii)} & \cosh \ell_{ij} = \dfrac{c_{ij}}{\sqrt{c_{ii} c_{jj}}}, \end{array}$$

where $c_{ij} = (-1)^{i+j} G_{ij}$ is ij-cofactor of Gram matrix G.

Theorem (Jacobi equation)

Let $G = (a_{ij})_{i,j=1,\dots,n}$ be an $n \times n$ matrix. Denote by $C = (c_{ij})_{i,j=1,\dots,n}$ the matrix of cofactors $c_{ij} = (-1)^{i+j} G_{ij}$, where G_{ij} is ij-th minor of matrix G. Then

$$\det (c_{ij})_{i,j=1,\dots,k} = \det G^{k-1} \cdot \det (a_{ij})_{i,j=k+1,\dots,n}.$$

Some facts about edge matrix of a hyperbolic tatrahedron

Theorem

Let T be a compact hyperbolic tetrahedron with **edge matrix** E*. Then*

$$\begin{aligned} &\text{(i)} && \det E > 0; \\ &\text{(ii)} && c_{ii} < 0,\ i \in \{1,2,3,4\}; \\ &\text{(iii)} && \cos\theta_{ij} = \frac{-c_{ij}}{\sqrt{c_{ii}c_{jj}}}, \end{aligned}$$

where $c_{ij} = (-1)^{i+j}E_{ij}$ is ij-cofactor of edge matrix E.

Our new formula in terms of edge matrix

Theorem (Abr. and B. Vuong, 2019)

Let T be a compact hyperbolic tetrahedron given by its **edge matrix** E *and* $c_{ij} = (-1)^{i+j} E_{ij}$ *is ij-cofactor of* E. *We assume that all the edge lengths are fixed exept* ℓ_{34} *which is formal variable. Then the volume* $V = V(T)$ *is given by the formula*

$$V = \frac{1}{2} \int\limits_{0}^{\ell_{34}} \frac{c_{14}c_{33}(c_{23}c_{44} - c_{24}c_{34}) + c_{13}c_{44}(c_{24}c_{33} - c_{23}c_{34})}{c_{33}c_{44} \det E \sqrt{c_{33}c_{44} - c_{34}^2}} \, t \sinh t \, dt,$$

where cofactors c_{ij} *and edge matrix determinant* $\det E$ *are functions in one variable* ℓ_{34} *denoted by* t.

To check if this formula is correct we put every edge lengths to be equal $\ell_{ij} = a$.

Particular case of a regular hyperbolic tetrahedron

Theorem (Abr. and B. Vuong, 2017 and many other works)

Let $T = T(a)$ be a regular hyperbolic tetrahedron and all of its edge lengths are equal to $a, a \geq 0$. Then the volume $V = V(T)$ is given by the formula

$$V = \int_0^a \frac{3\,t\ \sinh t\,dt}{(1+2\cosh t)\sqrt{(\cosh t+1)(3\cosh t+1)}}.$$

Thank you for attention!

- Proof of Sforza formula

We start with the the following theorem.

Theorem (Jacobi equation)

Let $G = (a_{ij})_{i,j=1,\dots,n}$ be an $n \times n$ matrix. Denote by $C = (c_{ij})_{i,j=1,\dots,n}$ the matrix of cofactors $c_{ij} = (-1)^{i+j} G_{ij}$, where G_{ij} is ij-th minor of matrix G. Then

$$\det{(c_{ij})_{i,j=1,\dots,k}} = \det{G}^{k-1} \cdot \det{(a_{ij})_{i,j=k+1,\dots,n}}.$$

New proof of Sforza formula

Apply the theorem to Gram matrix G for $n = 4$ and $k = 2$

$$G = \begin{pmatrix} 1 & -\cos\theta_{34} & x & x \\ -\cos\theta_{34} & 1 & x & x \\ x & x & x & x \\ x & x & x & x \end{pmatrix}, C = \begin{pmatrix} x & x & x & x \\ x & x & x & x \\ x & x & c_{33} & c_{34} \\ x & x & c_{34} & c_{44} \end{pmatrix}.$$

We have $\quad c_{33}c_{44} - c_{34}^2 = \det G \cdot (1 - \cos^2\theta_{34}) = \det G \cdot \sin^2\theta_{34}.$

From the other hand, by Cosine Rule

$$\cosh \ell_{34} = \frac{c_{34}}{\sqrt{c_{33}c_{44}}}, \quad \text{and} \quad \sinh \ell_{34} = \sqrt{\frac{c_{34}^2 - c_{33}c_{44}}{c_{33}c_{44}}},$$

Thus,

$$\sinh \ell_{34} = \frac{\sin\theta_{34}\sqrt{-\det G}}{\sqrt{c_{33}c_{44}}}.$$

New proof of Sforza formula

Since $\exp(\pm\ell_{34}) = \cosh\ell_{34} \pm \sinh\ell_{34}$ we have

$$\exp(\ell_{34}) = \frac{c_{34} + \sin\theta_{34}\sqrt{-\det G}}{\sqrt{c_{33}c_{44}}}, \quad \exp(-\ell_{34}) = \frac{c_{34} - \sin\theta_{34}\sqrt{-\det G}}{\sqrt{c_{33}c_{44}}}.$$

Hence,

$$\exp(2\ell_{34}) = \frac{c_{34} + \sin\theta_{34}\sqrt{-\det G}}{c_{34} - \sin\theta_{34}\sqrt{-\det G}}, \text{ and } \ell_{34} = \frac{1}{2}\log\frac{c_{34} + \sin\theta_{34}\sqrt{-\det G}}{c_{34} - \sin\theta_{34}\sqrt{-\det G}}.$$

By the Schläfli formula

$$-dV = \frac{1}{2}\sum_{ij}\ell_{ij}d\theta_{ij}, \quad i,j \in \{1,2,3,4\}$$

$$V = \int_{t_0}^{\theta_{34}}\left(-\frac{\ell_{34}}{2}\right)dt = \frac{1}{4}\int_{t_0}^{\theta_{34}}\log\frac{c_{34} - \sqrt{-\det G}\sin\theta_{34}}{c_{34} + \sqrt{-\det G}\sin\theta_{34}}dt,$$

where t_0 is a root of equation $\det G(t) = 0$.

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